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L. A. SKORNYAKOV

SYSTEMS OF LINEAR EQUATIONS

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

Л. А. Скорняков

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PREFACE

The subject matter of this booklet is stated quite clearly in the title. The theory of linear equations developed here is based solely on elementary transformations of matrices. Nowhere do I introduce the reader to the concept of complete induction, but in some places this concept is masked by "etc." Readers familiar with this concept will have no difficulty in bringing up the exposition to the current level of mathematical rigor. The main aim of the exercises is to give readers an understanding of how well they have understood the material. For a more detailed study of this topic readers are advised to turn to any course in linear algebra. Naturally, I find my book *Elements of Algebra* (Nauka, Moscow, 1980; in Russian) the most suitable for this purpose, but this opinion is purely subjective.

The idea at the base of the booklet has been used in teaching linear algebra in the Division of Structural Linguistics of the Department of Languages and Literature at Moscow State University. I am deeply grateful to Yu.A. Bakhturin for this idea. Yu.A. Bakhturin, D.V. Beklemishev, D.P. Egorov, and A.P. Mishina have given helpful remarks concerning the manuscript. I take the opportunity to thank all these mathematicians. I would also like to thank T.A. Gurova for her help in preparing the manuscript.

L.A. Skornyakov

1. SYSTEMS OF LINEAR EQUATIONS AND THEIR SOLUTIONS

A *linear equation* in n unknowns is an equation of the type

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b, \qquad (1.1)$$

where $a_1, a_2, \ldots, a_n$, and b are given real numbers. For instance,

$$2x_1 + x_2 = 3, \qquad (1.2)$$

$$x_1 + x_2 - x_3 = 0, \qquad (1.3)$$

and

$$x_1 + x_2 + x_3 + x_4 = 40 \qquad (1.4)$$

are equations in two, three, and four unknowns, respecetively. The numbers $a_1, a_2, \ldots, a_n$ are said to be th- *coefficients* of Eq. (1.1) and the number b is known as the *absolute term* of the same equation. What constitutes a solution to a linear equation? To answer this question, we introduce a row of length n,

$$(\alpha_1, \ldots, \alpha_n), \qquad (1.5)$$

where $\alpha_1, \ldots, \alpha_n$ are real numbers.* Note that the idea of a row is undefinable. Of course, one could say that a row is a sequence consisting of n real numbers. But then what is a sequence? Row (1.5) is said to be a *solution* to Eq. (1.1) if

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = b.$$

For instance, the following rows are solutions to Eq. (1.2): (1, 1), (0, 3), (2, —1). Other solutions can also be found. Among the solutions to Eq. (1.3) we have the rows (1, 1, 2), (1, 0, 1), and (0, 0, 0), while among the solutions to Eq. (1.4) we have (10, 10, 10, 10), (40, 0, 0, 0), and (70, —10, —10, —10). Any row of length n serves as a solution to the equation

$$0x_1 + 0x_2 + \ldots + 0x_n = 0,$$

while the equation

$$0x_1 + 0x_2 + \ldots + 0x_n = 1$$

---

* Often the term "vector" is used instead of "row".

has not a single solution. Thus, a linear equation may have many solutions. To solve such an equation means to describe in some fashion the entire set of solutions. For Eq. (1.2) the following description can be suggested: the solutions to Eq. (1.2) are the various rows of the form $(\alpha, 3 - 2\alpha)$, with α an arbitrary real number. The set of solutions to Eq. (1.3) consists of all possible rows of the type $(\alpha, \beta, \alpha + \beta)$, with α and β arbitrary real numbers. The solutions to Eq. (1.1) with $a_1 \neq 0$ are the rows

$$\left(\frac{b - a_2\alpha_2 - \ldots - a_n\alpha_n}{a_1}, \alpha_2, \ldots, \alpha_n\right),$$

where $\alpha_2, \ldots, \alpha_n$ are arbitrary real numbers. A similar description can be obtained for the case where some other coefficient of this equation is nonzero. But if all the coefficients are zeros, then for $b = 0$ any row of length n is a solution, while for $b \neq 0$ there is simply no solution.

Let us now turn to Eq. (1.4). Here is a problem that requires solving such an equation. Take a pool whose volume is 40 m³. Suppose the pool has four pipes through which the water is pumped into the pool. How much water must pass through each pipe for the pool to fill up? The solutions to Eq. (1.4) given above can be interpreted as follows: (1) 10 m³ of water must pass through each pipe, (2) 40 m³ is pumped through one pipe while the other three remain closed, and (3) 70 m³ is pumped into the pool through one pipe, while each of the other three pipes is used to pump 10 m³ of water out of the pool. The reader can easily see that three pipes out of the four can be used to pump an arbitrary amount of water in or out of the pool. However, if this is done, the amount of water that passes through the fourth pipe (in or out of the pool) is determined uniquely.

Let us now make the pool-pipe problem more complicated by introducing the condition that the amount of water passing through the third pipe must be equal to the amount of water passing through the other three pipes taken as a whole. Then the amount of water passing through each of the four pipes must satisfy, in addition to Eq. (1.4), the following equation

$$x_1 + x_2 - x_3 + x_4 = 0.$$

In this case it is usually said that the solution (the row) we are looking for must satisfy a system of two linear

equations

$$x_1 + x_2 + x_3 + x_4 = 40, \qquad (1.6)$$
$$x_1 + x_2 - x_3 + x_4 = 0.$$

Subtracting the second equation from the first, we find that $2x_3 = 40$, or $x_3 = 20$. Hence, in any case 20 m^3 of water must pass into the pool through the third pipe. The amount of water flowing through the other three pipes is governed by the equation

$$x_1 + x_2 + x_4 = 20,$$

which shows that through any two pipes an arbitrary amount of water can pass, but if this amount is fixed, the amount of water passing through the last pipe is fixed, too.

In the general case we must know how to deal with a system of m equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned} \qquad (1.7)$$

The row $(\alpha_1, \ldots, \alpha_n)$ is said to be a *solution* to this system if it is a solution to each of the equations in the system. For instance, among the solutions to Eq. (1.6) there are the following rows: (5, 5, 20, 10) (—15, 10, 20, 25) (there are also other solutions).

Note the notation in system (1.7). For instance, the subscripts in a_{12} stand for the fact that we are dealing with the second coefficient in the first equation. Therefore, one must say "a sub one two" and not "a sub twelve."

Even the simple system (1.6) illustrates the fact that the unknowns do not have an equal status. Some are determined by a system of linear equations, others can be fixed in an arbitrary manner, and still others are specified uniquely by the choice of the values of the unknowns that can be fixed arbitrarily. The aim of this book is to teach the reader how to carry out an appropriate analysis in each specific case.

Two systems of linear equations are said to be *equivalent* when each solution to one system is a solution to the other, and vice versa. It is clear that instead of a given system we can take (and solve) an equivalent one.

For instance, system (1.6) is equivalent to

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 40, \\
0x_1 + 0x_2 + x_3 + 0x_4 &= 20.
\end{aligned}
\tag{1.8}
$$

Indeed, if $\bar{u} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a solution to system (1.6), this row is a solution to the first equation in this system and, hence, a solution to the first equation in (1.8). Moreover, as noted earlier, α_3 must be equal to 20, which means that row $\bar{u}$ is a solution to the second equation in (1.8). Thus, every solution to system (1.6) is a solution to system (1.8). Now suppose that $\bar{v} = (\beta_1, \beta_2, \beta_3, \beta_4)$ is a solution to system (1.8). Just as before, we immediately see that $\bar{v}$ is a solution to the first equation in (1.6). Moreover, $\beta_3 = 20$ and $\beta_1 + \beta_2 + 20 + \beta_4 = 40$. This means that $\beta_1 + \beta_2 + \beta_4 = 20$ and, hence, $\beta_1 + \beta_2 - \beta_3 + \beta_4 = \beta_1 + \beta_2 - 20 + \beta_4 = 40 - 20 = 20$. Thus, $\bar{v}$ is a solution to the second equation in (1.6) and, hence, a solution to the entire system.

Theorem 1.1. *If to a system of linear equations in n unknowns we adjoin the equation*

$$0x_1 + 0x_2 + \ldots + 0x_n = 0,$$

the new system of equations is equivalent to the initial one.

▶* Since each equation of the old system is an equation of the new, each solution to the new system is a solution to the old. If row $\bar{u}$ is a solution to the old system, it is obviously a solution to all the equations in the new system with the exception, perhaps, of the adjoined equation. But $\bar{u}$ is a solution to the adjoined equation, too, since, as noted earlier, any row of length n can serve as solution to such an equation. Thus, each solution to the old system is a solution to the new. The proof is complete. ■**

Exercises

1. Set up systems of linear equations for solving the following problems:

(a) What are the lengths of the sides of a quadrangle if the sum of these lengths is 40 m and the sum of the lengths of the first three sides is by 20 m greater than the length of the fourth side?

---

* The symbol ▶ indicates the beginning of a proof.

** The symbol ■ indicates the end of a proof.

(b) In how many ways can you purchase something costing 2 roubles by using 20 coins with the following denominations: 5, 10, 15, and 20 kopecks (1 rouble equals 100 kopecks)?

(c) What values can four numbers have with the sum of any three of them being fixed at 1?

(d) What values can four numbers have with the sum of any two of them being fixed at 1?

(e) At a construction site there are four concrete mixers with a productive capacity of 20, 12, 15, and 10 tons of concrete per hour. The production quota for the four mixers is 120 tons of concrete every day in the course of three days. The conditions for the operation of the four mixers are as follows: (1) the mixer with the lowest productive capacity is in operation each day while the other three operate two days each, (2) only three operate simultaneously, and (3) the number of working hours is the same for each mixer in the course of the three days. How many hours will each mixer operate?

(f) What constitutes a sequence of n numbers if the sum of two adjacent members in the sequence is zero and so is the sum of the first and nth members?

2. Prove that if we adjoin the equation

$$(a_{11} + a_{21})\, x_1 + (a_{12} + a_{22})\, x_2 + \ldots + (a_{1n} + a_{2n})\, x_n = b_1 + b_2$$

to system (1.7), we arrive at a system equivalent to (1.7).

2. MATRICES AND THEIR ELEMENTARY TRANSFORMATIONS

The tool that makes it possible to solve the problem posed in the previous section is known as the matrix. Such mathematical objects emerge, as we will subsequently see, from rows of length n. The reader will recall that the idea of a row of length n is undefinable. If we have a row $\bar{u} = (a_1, \ldots, a_n)$, the numbers a_i are called the *coordinates*, or *components*, of the row. Two rows $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ are said to be *equal* if $m = n$ and $a_i = b_i$ for all i. A row consisting only of zeros is said to be a *zero row* and is denoted by $\bar{0}$. The *leader* in a row is the first nonzero coordinate in the row. For example, the leaders of the rows (0, 1, 0, 0, 0), (1, 0, 1), and (0, 0, 0, 0, 1) are in the second, first, and fifth places, respectively. Obviously, a zero row has no leader. Two rows of the same length can be added. The *sum* of two rows is defined as follows:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n).$$

Note that the "pluses" on the left- and right-hand sides have different meanings. On the left-hand side it designates the sum of two rows, while on the right-hand side it designates the sum of real numbers. For example,

$$(1, 2, 3, -1) + (2, 1, 0, 1) = (3, 3, 3, 0).$$

If $\bar{a}$ is an arbitrary row of length n and $\bar{0}$ is a zero row of length n, then, as it can easily be verified,

$$\bar{a} + \bar{0} = \bar{a} \quad \text{and} \quad \bar{0} + \bar{a} = \bar{a}.$$

Theorem 2.1. *If $\bar{a}$, $\bar{b}$, and $\bar{c}$ are arbitrary rows of length n, then*

$$\bar{a} + \bar{b} = \bar{b} + \bar{a}$$

and

$$(\bar{a} + \bar{b}) + \bar{c} = \bar{a} + (\bar{b} + \bar{c}).$$

▶ Let us assume that

$$\bar{a} = (\alpha_1, \ldots, \alpha_n)$$

and

$$\bar{b} = (\beta_1, \ldots, \beta_n).$$

Then

$$\bar{a} + \bar{b} = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$$

and

$$\bar{b} + \bar{a} = (\beta_1 + \alpha_1, \ldots, \beta_n + \alpha_n).$$

Since the commutative law is valid for addition of real numbers, we have

$$\alpha_1 + \beta_1 = \beta_1 + \alpha_1, \quad \alpha_2 + \beta_2 = \beta_2 + \alpha_2, \quad \ldots, \alpha_n + \beta_n = \beta_n + \alpha_n.$$

Thus, the rows $\bar{a} + \bar{b}$ and $\bar{b} + \bar{a}$ have the same length and the same ith coordinate for each i. By the definition of row equality, $\bar{a} + \bar{b} = \bar{b} + \bar{a}$, which is what we set out to prove. ■

To prove the second equality we put $\bar{c} = (\gamma_1, \ldots, \gamma_n)$ and note that the numbers $(\alpha_i + \beta_i) + \gamma_i$ and $\alpha_i + (\beta_i + \gamma_i)$ are the ith coordinates of the rows on the left- and right-hand sides of the equality. If we recall that the associative law holds true for real numbers, we will arrive at the sought equality.

Note that not all operations one encounters in mathematics are necessarily commutative. Take, for instance, the set of words written via the letters of the English alphabet. We will think of a word as a finite collection of such letters, say, *npk*, *mama*, *naotum*, and so on. We define the operation of addition as follows:

$$x_1 \ldots x_m + y_1 \ldots y_n = x_1 \ldots x_m y_1 \ldots y_n$$

(here x_i and y_i are not necessarily different letters).

It is clear that, say, $m + (ama) = mama \neq amam = (ama) + m$. Consequently, there are such words u and v that $u + v \neq v + u$. However, it can be proved that the equality $(u + v) + w = u + (v + w)$ holds for any words u, v, and w (carry out the proof). Another example of noncommutative operations is raising natural numbers to natural powers, since $2^3 \neq 3^2$.

Thanks to Theorem 2.1, we can add rows in the same way as we add numbers, that is, without paying attention to the order in which the members are added and placing the parentheses wherever we wish in adding several rows. Note that both for numbers and for rows the designation

$$\bar{a}_1 + \bar{a}_2 + \ldots + \bar{a}_m$$

has no meaning, strictly speaking, since we know how to add only two numbers or two rows. In calculating a "long" sum, we mentally place parentheses in this or that manner (for greater detail see *Elements of Algebra*, p. 36).

Besides addition, we will need to know how to multiply a row by a real number λ. This operation is defined as follows:

$$\lambda (a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda a_n).$$

Theorem 2.2. *If $\bar{a}$ and $\bar{b}$ are rows of length n, and λ is a real number, then*

$$\lambda (\bar{a} + \bar{b}) = \lambda \bar{a} + \lambda \bar{b},$$

$$(\lambda + \mu) \bar{a} = \lambda \bar{a} + \mu \bar{a},$$

$$(\lambda \mu) \bar{a} = \lambda (\mu \bar{a}),$$

and

$$1\bar{a} = \bar{a}.$$

▶ To prove, for instance, that $\lambda (\bar{a} + \bar{b}) = \lambda \bar{a} + \lambda \bar{b}$, we note that the ith coordinates of the rows on the left- and right-hand sides of the equality are $\lambda (a_i + b_i)$ and $\lambda a_i + \lambda b_i$, respectively, and their coincidence follows

from the fact that the distributive law holds true for real numbers. ■

The validity of the other equalities can be established by similar reasoning, and (we hope) the reader can now do all this by himself.

A table that consists of several rows of length n written one under the other is called a *matrix*. The matrix

$$\begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{Vmatrix}$$

contains m rows and n columns. We will say that this is a (rectangular) $m \times n$ matrix. Say, the matrices

$$\begin{Vmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix}, \quad \text{and} \quad \begin{Vmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{Vmatrix}$$

are 3×4, 4×3, and 2×5 matrices, respectively. A row of length n is simply a $1 \times n$ matrix, while a column of length (or height) m is an $m \times 1$ matrix. An $n \times n$ matrix is said to be a *square matrix of order n*. A matrix consisting of zero rows is called a *null matrix* (zero matrix) and is denoted by O, while the matrix

$$E = \begin{Vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{Vmatrix}$$

is known as the *identity matrix* (*unit matrix*). Note that for each number n there exists an identity matrix of order n, while in each set of $m \times n$ matrices there exists a null matrix. Two matrices are considered *equal* if they have the same number of rows and columns and their matrix elements with equal subscripts coincide.

As a rule, in what follows we will denote the elements of a matrix designated by a capital letter (say, A) by the corresponding small letter with the appropriate subscripts (in our case a_{ij}) if not specified otherwise. The first subscript designates the number of the row and the second the number of the column.

By a matrix transformation we mean a transition from one matrix to another carried out according to certain rules. We consider the following transformations:

(i) Interchanging two rows in a matrix.

(ii) Adding a row multiplied by a certain number to another row in the same matrix.

These transformations are known as *elementary transformations of rows of the first and second kinds*, respectively. Elementary transformations of columns can be defined in a similar manner.

Sometimes it is convenient to introduce the elementary transformation of the third kind:

(iii) Multiplication of a row by a nonzero number.

For example, take the matrix

$$\begin{Vmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{Vmatrix}.$$

Multiplying the first row by 2 and adding the result to the third row, we arrive at the matrix

$$\begin{Vmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 6 & -2 & 0 \end{Vmatrix}.$$

Now to the first row of this matrix multiplied by -2 we add the second row and arrive at

$$\begin{Vmatrix} 1 & 2 & -1 & 0 \\ 0 & -3 & 2 & 1 \\ 3 & 6 & -2 & 0 \end{Vmatrix}.$$

Theorem 2.3. *If in a matrix A the element a_{ik} is nonzero and $i \neq j$, then in the matrix B obtained from A by adding the ith row multiplied by $-a_{jk}/a_{ik}$ to the jth row the element b_{jk} is equal to zero.*

▶ From the definitions of the addition of rows and the multiplication of a row by a number we get (the row coordinates that do not appear in the formulas below are of no interest to us)

$$\begin{aligned}
&(b_{j1}, \ldots, b_{jk}, \ldots, b_{jn}) \\
&\quad = (a_{j1}, \ldots, a_{jk}, \ldots, a_{jn}) \\
&\quad + \left(-\frac{a_{jk}}{a_{ik}}\right)(a_{i1}, \ldots, a_{ik}, \ldots, a_{in}) \\
&\quad = (\ldots, a_{jk}, \ldots) + \left(\ldots, -\frac{a_{jk}}{a_{ik}} a_{ik}, \ldots\right) \\
&\quad = (\ldots, a_{jk}, \ldots) + (\ldots, -a_{jk}, \ldots) = (\ldots, 0, \ldots).
\end{aligned}$$

By virtue of the definition of row equality we then have $b_{jk} = 0$. ■

Theorem 2.4. *If going over from a matrix A to a matrix B requires a finite number of elementary transformations of rows, then going over from matrix B to matrix A requires a finite number of elementary transformations of rows, too.*

▶ We start by proving the following

Lemma. *If going over from a matrix A to a matrix B requires only one elementary transformation of rows, then going over from matrix B to matrix A also requires only one elementary transformation.*

Indeed, the validity of the lemma is obvious if an elementary transformation of the first kind is employed to transform A into B. But suppose an elementary transformation of the second kind is used to transform A into B, that is, (ith row of B) = (ith row of A) + λ (jth row of A), and all the other rows of B coincide with the respective rows of A. Thus, $b_{ik} = a_{ik} + \lambda a_{jk}$ for every k. If we now add the ith row of B to the jth row of B multiplied by $-\lambda$, the new matrix will have the following element with subscripts i and k:

$$b_{ik} + (-\lambda)\, b_{jk} = (a_{ik} + \lambda a_{jk}) + (-\lambda a_{jk}) = a_{ik}.$$

Since the elements of this new matrix that are situated in the rows with numbers distinct from i coincide with the appropriate elements of matrix A, the entire matrix coincides with A.

Suppose that A is transformed into B via t elementary transformations, with $t > 1$. By $C_1, C_2, \ldots, C_{t-1}$ we denote the matrices obtained successively in such transformations. Thus, using only one elementary transformation, we can go from A to C_1, using another elementary transformation we go from C_1 to C_2, and so on. In this manner we go from C_i to C_{i+1} for $i = 1, 2, \ldots, t-2$, and then from C_{t-1} to B. But according to the above lemma, we need only one elementary transformation to go from B to C_{t-1}, one to go from C_{t-1} to C_{t-2}, etc. We will finally arrive at matrix C_1, from which we move on to A. The proof of the theorem is complete. ■

We call a matrix a *step-like matrix* if it possesses the following properties:

(1) If the ith row is a zero row, then the $(i+1)$st row is a zero row, too.

(2) If the leaders of the ith and the $(i+1)$st row are in the k_ist and k_{i+1}st columns, respectively, then

$$k_i < k_{i+1}.$$

Graphically these properties mean that only zero rows can lie below a zero row, while all the elements that lie to the left and below the leader of a row are zeros. The origin of the name can easily be seen from the following step-like matrix:

$$\begin{Vmatrix} 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix},$$

with $k_1 = 2$, $k_2 = 4$, and $k_3 = 5$.

Theorem 2.5. *If A is a step-like $m \times n$ matrix and B is obtained from A by adjoining to A from below a number of zero rows of length n, then B is a step-like matrix.*

▶ Suppose that the rth row of matrix B is a zero row. If this row belongs to those that were adjoined to A, then all the rows that are below it are zero rows by hypothesis. But if the rth row belongs to matrix A, then according to the definition of a step-like matrix the rows of matrix A that are below the rth row must be zero rows, too. Below these zero rows of A are the adjoined rows, which are zero rows by hypothesis. Hence, matrix B satisfies the first property of a step-like matrix. Since all the nonzero rows of matrix B are also rows of matrix A and are arranged in the same order, their leaders satisfy the second property of a step-like matrix. This means that B is a step-like matrix. ■

The following theorem plays an important role in the construction of the theory we are seeking.

Theorem 2.6. *Every matrix may be converted to a step-like matrix by a finite number of elementary transformations of the rows.*

▶ Suppose that A is an arbitrary matrix and m is the number of rows in this matrix. If $A = 0$, the matrix is step-like. If A is not a null matrix, it must have at least one nonzero element. A nonzero element must belong to a row, so that our matrix has nonzero rows. Out of the nonzero rows we select the one whose leader is in the column with the lowest number, say k_1. Applying the

elementary transformation of the first kind, we interchange this row with the first row. The matrix then assumes the form

$$\begin{Vmatrix} 0 & \dots & 0 & b_{1k_1} & \dots \\ 0 & \dots & 0 & b_{2k_1} & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & b_{mk_1} & \dots \end{Vmatrix},$$

with $b_{1k_1} \neq 0$. Now we apply the transformations of the second kind, namely, the first row is multiplied by $-b_{2k_1}/b_{1k_1}$ and the product is added to the second row, the first row is multiplied by $-b_{3k_1}/b_{1k_1}$ and the product is added to the third row, and so on. By Theorem 2.4, after applying $m - 1$ such elementary transformations we arrive at a matrix whose elements in the k_1st column are all zeros except the first element:

$$\begin{Vmatrix} 0 & \dots & 0 & b_{1k_1} & \dots \\ 0 & \dots & 0 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & \dots \end{Vmatrix}.$$

The row matrix

$$\| \; 0 \; \dots \; 0 \; b_{1k_1} \; \dots \; \|$$

is step-like, since the conditions that define a step-like matrix are satisfied for this matrix in a trivial manner: there are no zero rows and there is no row with the leader in the k_{i+1}st column. By Theorem 2.5, matrix B proves to be step-like if all the rows starting from the second are zero rows. If this is not so, then, just as in the case above, among the rows of B we find the one whose leader is in the column with the lowest number, say k_2. Since to the left of and below element b_{1k_1} there are only zeros, we have $k_1 < k_2$. We put the row we have found in the second place and, using Theorem 2.4 once more, we arrive at the matrix

$$C = \begin{Vmatrix} 0 & \dots & 0 & b_{1k_1} & b_{1k_1+1} & \dots & b_{1k_2-1} & b_{1k_2} & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & c_{2k_2} & \dots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots \end{Vmatrix},$$

where $c_{2k_2} \neq 0$ (of course, it may so happen that $k_2 = k_1 + 1$ and elements $b_{1k_1+1}, \ldots, b_{1k_2-1}$ will simply not exist). It is clear that the first two rows of matrix C constitute a step-like matrix. If $m = 2$ or all the rows of matrix C starting from the second are zero rows, then, as in the above case, C is step-like. But if among these rows there are nonzero rows, the same line of reasoning as above brings us to the following matrix:

$$\begin{Vmatrix} 0 \ldots 0 & b_{1k_1} & b_{1k_1+1} & \ldots & b_{1k_2-1} & b_{1k_2} & b_{1k_2+1} & \ldots & b_{1k_3-1} & b_{1k_3} & \ldots \\ 0 \ldots 0 & 0 & 0 & \ldots & 0 & c_{2k_2} & c_{2k_2+1} & \ldots & c_{2k_3-1} & c_{2k_3} & \ldots \\ 0 \ldots 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & d_{2k_3} & \ldots \\ 0 \ldots 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 \ldots 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \end{Vmatrix},$$

where $d_{2k_3} \neq 0$. We note again that the matrix consisting of the first three rows is step-like, and either the entire matrix is step-like or it can be converted into a matrix whose first four rows constitute a step-like matrix. It is clear that the number of transformations needed to arrive at a step-like matrix containing all m rows can be no greater than m. ■

The proof of Theorem 2.6 is effective in the sense that it contains a practical method for reducing a given matrix to step-like form. For example, reduction of the matrix

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & 1 & 0 & 2 & 4 \\ 1 & 1 & -1 & 3 & 4 \end{Vmatrix}$$

to step-like form can be carried out in the following manner:

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{Vmatrix}$$ (the first row is subtracted from the second, third and fourth rows),

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$ (to the third row we add the second, and to the fourth we add the second multiplied by 2).

With matrix

$$\begin{Vmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{Vmatrix}$$

we proceed as follows;

$$\begin{Vmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{Vmatrix}$$ (the first and second rows are interchanged),

$$\begin{Vmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{Vmatrix}$$ (the first row is subtracted from the second),

$$\begin{Vmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$ (the third row is added to the second).

Finally, for the matrix

$$\begin{Vmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 4 & 9 & 14 \end{Vmatrix}$$

we have

$$\begin{Vmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 11 \end{Vmatrix}$$ (the first row is subtracted from the second and third rows),

and

$$\begin{Vmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{Vmatrix}$$ (the second row multiplied by -3 is added to the third row).

Exercises

1. Suppose that the leaders of the rows $\bar{a}$, $\bar{b}$, and $\bar{a}+\bar{b}$ are in the kth, lth, and mth columns, respectively, with $k \leqslant l$. Prove that $k \leqslant m$. Give examples of rows for which $k < m$. Finally, prove that this is possible only for $k = l$.

2. Suppose that $\bar{a} = (1, 2, -3, 0, 1)$ and $\bar{b} = (1, -1, 0, 2, 3)$. Find the row $\bar{x}$ for which $\bar{a} + \bar{x} = \bar{b}$.

3. Prove that the equation $\bar{a} + \bar{x} = \bar{0}$ has a solution for every row $\bar{a}$.

4. Prove that $0\bar{a} = \bar{0} = \lambda\bar{0}$ and $(-1)\,\bar{a} + \bar{a} = \bar{0}$ for every row $\bar{a}$ and every real number λ.

5. Suppose that $\bar{a} = (5, -8, -1, 2)$, $\bar{b} = (2, -1, 4, -3)$, and $\bar{c} = (-3, 2, -5, 4)$. Find the rows $\bar{x}$ and $\bar{y}$ that satisfy the equations

$$\bar{a} + 2\bar{b} + 3\bar{c} + 4\bar{x} = \bar{0}$$

and

$$3(\bar{a}-\bar{y})+2(\bar{a}+\bar{y})=5(\bar{c}+\bar{y}).$$

6. It is said that a row $\bar{a}$ is a *linear combination* of rows $\bar{a}_1, \ldots$ $\ldots, \bar{a}_m$ if $\bar{a}=\lambda_1\bar{a}_1+\ldots+\lambda_m\bar{a}_m$, where $\lambda_1, \ldots, \lambda_m$ is a set of real numbers.

(a) Find the linear combination $3\bar{a}+5\bar{b}-\bar{c}$, where $\bar{a}=(4, 1, 3, -2)$, $\bar{b}=(1, 2, -3, 2)$, and $\bar{c}=(16, 9, 1, -3)$.

(b) If a row $\bar{c}$ is a linear combination of rows $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s$ and each of these rows is a linear combination of rows $\bar{a}_1, \bar{a}_2, \ldots$ $\ldots, \bar{a}_r$, then $\bar{c}$ is a linear combination of the rows $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r$. Prove this assertion.

7. Reduce the following matrices to step-like form:

(a) $\left\|\begin{matrix} 1 & 1 & 1 & 1 & 40 \\ 1 & 1 & 1 & -1 & 20 \end{matrix}\right\|$; (b) $\left\|\begin{matrix} 1 & 1 & 1 & 1 & 20 \\ 5 & 10 & 15 & 20 & 200 \end{matrix}\right\|$;

(c) $\left\|\begin{matrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{matrix}\right\|$; (d) $\left\|\begin{matrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{matrix}\right\|$;

(e) $\left\|\begin{matrix} 20 & 12 & 0 & 10 & 120 \\ 20 & 0 & 15 & 10 & 120 \\ 0 & 12 & 15 & 10 & 120 \end{matrix}\right\|$;

(f) $\left\|\begin{matrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{matrix}\right\|$

(the last matrix is an $n \times n$ matrix).

8. Prove that the ith and jth rows of a matrix can be interchanged through the following sequence of elementary transformations: (1) multiply the jth row by -1 and add the product to the ith row, (2) add the ith row to the jth row, (3) add the product of the jth row and -1 to the ith row, and (4) multiply the ith row by -1.

9. Prove that any matrix can be reduced to diagonal form by elementary transformations of the rows and columns (a matrix D is said to be *diagonal* if and only if $i \neq j$ implies $d_{ij}=0$). Give an example of a matrix that cannot be reduced to diagonal form solely by elementary transformations of rows.

10. Prove the analog of Theorem 2.4 for elementary transformations of the third kind and the analogs of Theorems 2.4 and 2.6 for elementary transformations of columns.

11. Prove that by applying elementary transformations of the second kind to the rows and columns of matrix $A = \begin{Vmatrix} a & 0 \\ 0 & b \end{Vmatrix}$ we can reduce it to diagonal form $\begin{Vmatrix} 1 & 0 \\ 0 & -ab \end{Vmatrix}$.

12. Suppose that a matrix B can be obtained from a matrix A via a finite number of elementary transformations of the first, second, and third kinds applied to the rows. Prove that each row of matrix B is a linear combination of the rows of matrix A. *Hint.* Use the result of Exercise 6*b*.

13. (A.N. Mironov). Suppose that B and C are step-like matrices obtained from a matrix A via a finite number of elementary transformations of the first, second, and third kinds applied to the rows of A. Prove that leaders of the rows of matrices B and C are in the same columns. *Hint.* Use the result of Exercise 12.

3. A METHOD FOR SOLVING SYSTEMS OF LINEAR EQUATIONS

A system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{3.1}$$

is determined uniquely by the matrix

$$\left\Vert \begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array} \right\Vert,$$

which is known as the *augmented matrix of the system*. The matrix to the left of the single vertical bar is called the *system matrix*. For instance, for the systems

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4, \\ x_1 + x_2 + 2x_3 \qquad &= 4, \\ x_1 + x_2 \qquad + 2x_4 &= 4, \\ x_1 + x_2 - x_3 + 3x_4 &= 4; \end{aligned} \tag{3.2}$$

$$\begin{aligned} -x_3 + x_4 &= 1, \\ x_2 + 2x_3 - x_4 &= 0, \\ x_2 + x_3 \qquad &= 0, \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 3,\\
x_1 + 2x_2 + 3x_3 &= 6,\\
x_1 + 4x_2 + 9x_3 &= 14
\end{aligned}
\tag{3.4}
$$

the augmented matrices are

$$
\left\|\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4\\ 1 & 1 & 2 & 0 & 4\\ 1 & 1 & 0 & 2 & 4\\ 1 & 1 & -1 & 3 & 4 \end{array}\right\|, \quad \left\|\begin{array}{cccc|c} 0 & 0 & -1 & 1 & 1\\ 0 & 1 & 2 & -1 & 0\\ 0 & 1 & 1 & 0 & 0 \end{array}\right\|,
$$

and

$$
\left\|\begin{array}{cccc} 1 & 1 & 1 & 3\\ 1 & 2 & 3 & 6\\ 1 & 4 & 9 & 14 \end{array}\right\|,
$$

respectively.

In what follows, when we speak of the rows of a system of linear equations we mean the rows of the corresponding augmented matrix.

Theorem 3.1. *If a row in a system of linear equations is multiplied by a nonzero number, the resulting system of linear equations is equivalent to the initial one.*

▶ Suppose that a nonzero number λ is multiplied by the ith row of system (3.1). The corresponding equation assumes the form

$$(\lambda a_{i1})\, x_1 + (\lambda a_{i2})\, x_2 + \ldots + (\lambda a_{in})\, x_n = \lambda b_i. \tag{3.5}$$

If $(\alpha_1, \ldots, \alpha_n)$ is a solution to system (3.1), we substitute it into the left-hand side of Eq. (3.5) and get

$$
\begin{aligned}
&(\lambda a_{i1})\, \alpha_1 + (\lambda a_{i2})\, \alpha_2 + \ldots + (\lambda a_{in})\, \alpha_n\\
&\qquad = \lambda\, (a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots + a_{in}\alpha_n) = \lambda b_i.
\end{aligned}
$$

Hence, the row $(\alpha_1, \ldots, \alpha_n)$ is a solution to Eq. (3.5). Since the other equations in the new system are the same as in the old, the row is a solution to the new system. Now suppose that the row $(\alpha_1, \ldots, \alpha_n)$ is a solution to the new system. Then it is a solution to all the equations of system (3.1) with the exception, perhaps, of the ith equation. Substituting this solution into Eq. (3.5) and taking λ out of the parentheses, we get

$$\lambda\, (a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots + a_{in}\alpha_n) = \lambda b_i.$$

Since $\lambda \neq 0$, we find that

$$a_{i1}\alpha_1 + a_{i2}\alpha_2 + \ldots + a_{in}\alpha_n = b_i,$$

that is, $(\alpha_1, \ldots, \alpha_n)$ proves to be a solution to the ith equation in (3.1), which means it is a solution to the entire system (3.1). ■

Theorem 3.2. *If going over from a matrix $\tilde{A}$ to a matrix $\tilde{B}$ requires only a finite number of elementary transformations of rows, the corresponding systems of linear equations are equivalent.*

We start by proving the following

Lemma. *If going over from matrix $\tilde{A}$ to matrix $\tilde{B}$ requires only a finite number of elementary transformations of rows, then each solution to the system corresponding to matrix $\tilde{A}$ is a solution to the system corresponding to matrix $\tilde{B}$.*

▶ Let us first assume that going over from $\tilde{A}$ to $\tilde{B}$ requires only one elementary transformation. If this transformation is of the first kind, that is, two rows are interchanged, our equations in the system only trade places. Naturally, the old solutions will still satisfy such a system. Under an elementary transformation of the second kind, the jth row is multiplied by λ and the product is added to the ith row. Hence, the ith row of matrix $\tilde{B}$ has the form

$$(a_{i1} + \lambda a_{j1}, \ldots, a_{in} + \lambda a_{jn} \mid b_i + \lambda b_j).$$

Suppose that $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a solution to the system with matrix $\tilde{A}$ (the augmented matrix), that is, it is a solution to each equation in this system. But will it be a solution to the system with matrix $\tilde{B}$? Only the ith equation in this system may cause trouble, but

$$\begin{aligned}(a_{i1} + \lambda a_{j1})\, \alpha_1 + \ldots + (a_{in} + \lambda a_{jn})\, \alpha_n \\ = (a_{i1}\alpha_1 + \ldots + a_{in}\alpha_n) \\ + \lambda\, (a_{j1}\alpha_1 + \ldots + a_{jn}\alpha_n) = b_i + \lambda b_j.\end{aligned}$$

Thus, for the case at hand the proof of the lemma is complete. ■

In the general case we have a sequence of matrices $\tilde{A}$, $\tilde{C}_1, \ldots, \tilde{C}_k$, $\tilde{B}$, where at each step going over from a matrix to the adjacent right matrix requires only one elementary transformation. For this reason a solution to a system whose augmented matrix is $\tilde{A}$ serves as a solution to a system whose augmented matrix is $\tilde{C}_1$,

But then it also serves as a solution to a system whose augmented matrix is $\widetilde{C}_2$. Proceeding in this manner, we finally arrive at the conclusion that this solution is a solution to the system whose augmented matrix is $\widetilde{B}$.

▶ Note that according to the lemma, each solution to the system corresponding to matrix $\widetilde{A}$ is also a solution to the system corresponding to matrix $\widetilde{B}$. On the other hand, by Theorem 2.4, going over from matrix $\widetilde{B}$ to matrix $\widetilde{A}$ requires only elementary transformations. Hence, applying the lemma once more, we will see that each solution to the system whose augmented matrix is $\widetilde{B}$ serves as a solution to the system whose augmented matrix is $\widetilde{A}$. Thus, these systems are equivalent. ■

Now it is clear that to find the solutions to any system of linear equations it is sufficient to know how to find the solutions to a step-like system, since each matrix can be reduced to step-like form via elementary transformations, with the result that we arrive at a system of equations equivalent to the initial system.

For example, noting that at the end of Section 2 we reduced the augmented matrices of the systems (3.2)-(3.4) to step-like form, we conclude that instead of these systems we can solve the systems

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4, \\ x_3 - x_4 &= 0 \end{aligned}$$

(the last two equations have been discarded on the basis of Theorem 1.1),

$$\begin{aligned} x_2 + x_3 \phantom{{}- x_4} &= 0, \\ x_3 - x_4 &= 0, \\ 0x_1 + 0x_2 + 0x_3 - 0x_4 &= 1, \end{aligned}$$

and

$$\begin{aligned} x_1 + x_2 + x_3 &= 3, \\ x_2 + 2x_3 &= 3, \\ 2x_3 &= 2, \end{aligned}$$

respectively.

Now let us analyze step-like systems of linear equations. Suppose we have a step-like matrix corresponding to a system of m linear equations in n unknowns. Two cases are possible:

(i) There is a row whose leader is in the last column.
(ii) There is no such row.

In the first case the appropriate system of equations contains an equation of the form $0x_1 + \ldots + 0x_n = b$, where $b \neq 0$. It is clear there is not a single set of values of the x_i that can satisfy this equation, to say nothing of all the equations in the system. This means that the system of equations has no solution.

To analyze the second case we assume that the step-like matrix under consideration contains r nonzero rows and that the first nonzero elements in these rows are in the columns with numbers $k_1, \ldots, k_r$. By the very definition of a step-like matrix,

$$1 \leqslant k_1 < k_2 < \ldots < k_r \leqslant n.$$

The unknowns $x_{k_1}, \ldots, x_{k_r}$ are called *principal unknowns*, while the rest (if such exist) are called *absolute unknowns*. Moreover, we discard the equations that correspond to zero rows, which procedure will lead to a system that is equivalent to the initial one, according to Theorem 1.1.

Let us first assume that no absolute unknowns exist. Then

$$r = n, \quad k_1 = 1, \quad k_2 = 2, \ldots, k_n = n,$$

and the system we are considering here has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n-1}x_{n-1} + a_{1n}x_n &= b_1, \\ a_{22}x_2 + \ldots + a_{2\,n-1}x_{n-1} + a_{2n}x_n &= b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n &= b_{n-1}, \\ a_{nn}x_n &= b_n, \end{aligned}$$

with $a_{11}, a_{22}, \ldots, a_{nn}$ being nonzero. Since $a_{nn} \neq 0$, the last equation provides a unique way for determining x_n. After this, using the next-to-last equation, we can find x_{n-1}. The process can be continued. Thus, in the case at hand the system has a unique solution.

Now suppose that we do have absolute unknowns in our system. In this case, by L_i we denote the sum of the products of the absolute unknowns by the corresponding coefficients of these unknowns in the ith equation. Carrying the L_i to the right-hand sides of the equations in the

system, we arrive at

$$\begin{aligned} a_{1k_1}x_{k_1}+a_{1k_2}x_{k_2}+\dots+a_{1k_r}x_{k_r} &= b_1-L_1,\\ a_{2k_2}x_{k_2}+\dots+a_{2k_r}x_{k_r} &= b_2-L_2,\\ \dots\dots\dots\dots\dots &\\ a_{rk_r}x_{k_r} &= b_r-L_r, \end{aligned}$$

where the coefficients a_{1k_1}, a_{2k_2}, . . ., a_{rk_r} are nonzero. Just as in the case without absolute unknowns, we can determine x_{k_r}, $x_{k_{r-1}}$, etc. successively if we assign definite values to the absolute unknowns. The principal unknowns are then determined uniquely. Assigning different values to the absolute unknowns, we get all the solutions to the given system, or solve it. Since the absolute unknowns can assume different values, the system has more than one solution.

Analyzing in this manner the step-like systems that emerge from systems (3.2)-(3.4), we conclude that system (3.3) has no solutions, system (3.4) has a unique solution (1, 1, 1), and in system (3.2) x_1 and x_3 are the principal unknowns and x_2 and x_4 are the absolute unknowns, with $x_3 = x_4$ and $x_1 = 4 - x_2 - 2x_4$. The last statement can be interpreted as follows: each solution to system (3.2) has the form

$$(4-\alpha-2\beta,\ \alpha,\ \beta,\ \beta),$$

where α and β are arbitrary real numbers.

A system of linear equations is said to be *homogeneous* if all its absolute terms are zeros (all equations are homogeneous). A homogeneous system always has a solution, say, the zero row. It would be interesting to know, therefore, when a homogeneous system has nonzero solutions.

Theorem 3.3. *If the number of equations in a homogeneous system of linear equations is less than the number of the unknowns, the system has at least one nonzero solution.*

▶ We reduce the given homogeneous system to step-like form. In the process of such a transformation the system remains homogeneous. It is clear that the number of principal unknowns cannot be greater than the number of rows. This means that there are absolute unknowns, which ensures the existence of nonzero solutions, since absolute unknowns can assume nonzero values. ■

Theorem 3.4. *A homogeneous step-like system of n linear equations in n unknowns has nonzero solutions if and only if its matrix contains a zero row.*

▶ If there is a zero row, we can discard it, according to Theorem 1.1, and the existence of nonzero solutions follows from Theorem 3.3. But if there is not a single zero row in the system, all the unknowns are principal unknowns, and since all the absolute terms are zeros, we find (successively, starting from the end) that all the unknowns must be zeros. ■

Theorem 3.5. *If the rows $\bar{u}_1, \ldots, \bar{u}_t$ are solutions to the equation*

$$a_1x_1 + \ldots + a_nx_n = 0, \tag{3.6}$$

then for any real numbers $\lambda_1, \ldots, \lambda_t$ the row

$$\bar{u} = \lambda_1\bar{u}_1 + \ldots + \lambda_t\bar{u}_t$$

is also a solution to this equation.

▶ Suppose that

$$\bar{u}_i = (\alpha_{i1}, \ldots, \alpha_{in}) \quad (i = 1, 2, \ldots, t).$$

Then

$$\bar{u} = (\lambda_1\alpha_{11} + \ldots + \lambda_t\alpha_{t1}, \ldots, \lambda_1\alpha_{1n} + \ldots + \lambda_t\alpha_{tn}).$$

Since the $\bar{u}_i$ are solutions to Eq. (3.6), we have

$$\begin{aligned} a_1(\lambda_1\alpha_{11} + \ldots + \lambda_t\alpha_{t1}) & \\ + \ldots + a_n(\lambda_1\alpha_{1n} + \ldots + \lambda_t\alpha_{tn}) & \\ = \lambda_1(a_1\alpha_{11} + \ldots + a_n\alpha_{1n}) & \\ + \ldots + \lambda_t(a_1\alpha_{t1} + \ldots + a_n\alpha_{tn}) & \\ = \lambda_1 \cdot 0 + \ldots + \lambda_t \cdot 0 = 0, & \end{aligned}$$

that is, $\bar{u}$ proves to be a solution to Eq. (3.6), too. ■

Summarizing, we can say that we have a method that enables solving any system of linear equations, that is, we can establish that the system has no solutions or find the unique solution or, specifying the absolute unknowns, express the other unknowns in terms of them. This method is known as *Gauss's method* or the *method of successive elimination of unknowns.* But does the number of absolute unknowns depend on the method of solution? More than that, even in the case where no such dependence exists, can we guarantee that for a given set of

unknowns there exists a method of solution in which precisely these unknowns are absolute? The reader will recall that in Section 1 we gave an example of a system with absolute unknowns where, however, not every unknown could be called absolute. And the importance of knowing the number of the unknowns that can be declared absolute cannot be overestimated. Suppose that a technological problem requires solving a system of linear equations. If an unknown cannot be declared absolute, then it cannot be varied to achieve a more acceptable, from the practical viewpoint, solution. For instance, in the pool-pipe problem discussed in Section 1, additional restrictions can be imposed only on the amount of water passing through the first, second, and fourth pipes.

An answer to all these questions is given by a theory we will develop in the following sections.

Exercises

1. Solve the following systems of linear equations:

(a)
$$\begin{aligned} 5x_1 + 3x_2 + 5x_3 + 12x_4 &= 10, \\ 2x_1 + 2x_2 + 3x_3 + 5x_4 &= 4, \\ x_1 + 7x_2 + 9x_3 + 4x_4 &= 2. \end{aligned}$$

(b)
$$\begin{aligned} -9x_1 + 6x_2 + 7x_3 + 10x_4 &= 3, \\ -6x_1 + 4x_2 + 2x_3 + 3x_4 &= 2, \\ -3x_1 + 2x_2 + 11x_3 - 15x_4 &= 1. \end{aligned}$$

(c)
$$\begin{aligned} 9x_1 - 20x_2 + 3x_3 + 7x_4 &= 1, \\ 4x_1 - 9x_2 + x_3 + 2x_4 &= 2, \\ -2x_1 + 5x_2 + x_3 + 4x_4 &= 8. \end{aligned}$$

(*Hint*. To simplify calculations, the second row can be multiplied by two and the product subtracted from the first row.)

(d)
$$\begin{aligned} 12x_1 + 9x_2 + 3x_3 + 10x_4 &= 13, \\ 4x_1 + 3x_2 + x_3 + 2x_4 &= 3, \\ 8x_1 + 6x_2 + 2x_3 + 5x_4 &= 7. \end{aligned}$$

(e)
$$\begin{aligned} -6x_1 + 9x_2 + 3x_3 + 2x_4 &= 4, \\ -2x_1 + 3x_2 + 5x_3 + 4x_4 &= 2, \\ -4x_1 + 6x_2 + 4x_3 + 3x_4 &= 3. \end{aligned}$$

(f)
$$\begin{aligned} x_1 + x_2 + 4x_3 - 3x_4 &= 0, \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0, \\ 4x_1 + 5x_2 + 2x_3 + 3x_4 &= 0, \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0. \end{aligned}$$

(g)
$$\begin{aligned}
x_1 - x_3 &= 0,\\
x_2 - x_4 &= 0,\\
-x_1 + x_3 - x_5 &= 0,\\
-x_2 + x_4 - x_6 &= 0,\\
-x_3 + x_5 &= 0,\\
-x_4 + x_6 &= 0.
\end{aligned}$$

(h)
$$\begin{aligned}
x_1 - x_3 + x_5 &= 0,\\
x_2 - x_4 + x_6 &= 0,\\
x_1 - x_2 + x_5 - x_6 &= 0,\\
x_2 - x_3 + x_6 &= 0,\\
x_1 - x_4 + x_5 &= 0.
\end{aligned}$$

(i)
$$\begin{aligned}
5x_1 + 6x_2 - 2x_3 + 7x_4 + 4x_5 &= 0,\\
2x_1 + 3x_2 - x_3 + 4x_4 + 2x_5 &= 0,\\
7x_1 + 9x_2 - 3x_3 + 5x_4 + 6x_5 &= 0,\\
5x_1 + 9x_2 - 3x_3 + x_4 + 6x_5 &= 0.
\end{aligned}$$

(j)
$$\begin{aligned}
3x_1 + 4x_2 + x_3 + 2x_4 + 3x_5 &= 0,\\
5x_1 + 7x_2 + x_3 + 3x_4 + 4x_5 &= 0,\\
4x_1 + 5x_2 + 2x_3 + x_4 + 5x_5 &= 0,\\
7x_1 + 10x_2 + x_3 + 6x_4 + 5x_5 &= 0.
\end{aligned}$$

(k)
$$\begin{aligned}
x_1 + x_2 \qquad\quad &= 0,\\
x_1 + x_2 + x_3 &= 0,\\
x_2 + x_3 + x_4 &= 0,\\
\cdots\cdots\cdots\cdots&\cdots\\
x_{n-2} + x_{n-1} + x_n &= 0,\\
x_{n-1} + x_n &= 0.
\end{aligned}$$

2. Solve the problems formulated in Exercise 1 of Section 1. In Problem (b) find the solution under the condition that the number of 20-kopeck and 15-kopeck coins must be maximal. In Problem (e) find the solutions corresponding to the maximal and minimal working hours of the mixer with the lowest productive capacity (the number of working hours must be an integer).

3. Prove that a system of linear equations is homogeneous if and only if a zero row is a solution to this system.

4. Construct a system consisting of two linear equations for which the rows (1, 1, 1, 1) and (1, 2, 2, 1) are solutions.

5. Prove that by discarding a row of the augmented matrix of a system of linear equations that is a linear combination of the other rows we get a system of linear equations equivalent to the given system.

6. Formulate and prove the conditions imposed on the elements of the augmented matrix of a system of linear equations in n unknowns that are necessary and sufficient for every row of length n to be a solution to this system.

7. Can a system of linear equations with real coefficients have exactly two distinct solutions?

4. THE RANK OF A MATRIX

A square matrix is said to be *singular* if a finite number of elementary transformations of rows transforms it into a matrix with at least one zero row.

Theorem 4.1. *The following properties of a square matrix A are equivalent:*

(1) *A is a singular matrix;*

(2) *the system of homogeneous linear equations with matrix A has at least one nonzero solution;*

(3) *for any method that reduces matrix A to step-like form via elementary transformations of rows, the resulting step-like matrix contains a zero row.*

▶ The theorem states that if one of the properties (1)-(3) is valid, so are the other two. To establish this, we will prove that property (2) follows from property (1), property (3) from property (2), and property (1) from property (3) (Fig. 1), from which follows the sought equivalence of all three properties.

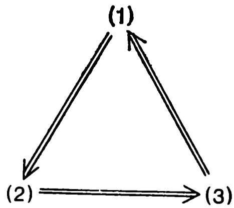

Fig. 1

In the course of the entire proof we denote the matrix obtained from A by adjoining the zero column on the right by $\bar{A}$.

(1) $\Rightarrow$ (2). If A is a singular matrix, then, according to the definition of a singular matrix, a finite number of elementary transformations of rows transforms A into a matrix B with a zero row. The same elementary transformations transform $\bar{A}$ into $\bar{B}$. Matrix $\bar{B}$ then contains a zero row, and if we discard this row we arrive, according to Theorem 1.1, at a system that is equivalent to a system with augmented matrix $\bar{B}$. But the number of equations in such a system is lower than the number of unknowns, and by Theorem 3.3 it has a nonzero solution. Hence, the system with augmented matrix $\bar{B}$ also has a nonzero

solution, and the same is true of the system with matrix $\overline{A}$ because of Theorem 3.2.

(2) $\Rightarrow$ (3). Suppose that C is the step-like matrix obtained from A via a finite number of elementary transformations. Then matrix $\overline{C}$ is obtained from $\overline{A}$ via the same elementary transformations. Theorem 3.2 implies that the system with augmented matrix $\overline{C}$ has a nonzero solution. By Theorem 3.4, this means that matrix $\overline{C}$ has a zero row, and so does matrix C.

(3) $\Rightarrow$ (1). It is sufficient to recall that by Theorem 2.6 matrix A can be reduced to step-like form and to employ property (3). ■

Theorem 4.2. *If A is a singular matrix and a finite number of elementary transformations is needed to go over from A to B, then B is a singular matrix, too.*

▶ By Theorem 2.4, going over from matrix B to matrix A requires a finite number of elementary transformations of rows. On the other hand, by the very definition of a singular matrix, a finite number of appropriate elementary transformations of rows makes it possible to go over from matrix A to a matrix C with a zero row. Hence, going over from B to C also requires a finite number of elementary transformations of rows, and the fact that B is singular follows from the definition. ■

Theorem 4.3. *If A is a singular matrix and a matrix B is obtained from matrix A through multiplication of one of the rows of A by a real number λ, then B is singular, too.*

▶ If $\lambda = 0$, matrix B is singular by definition. But if $\lambda \neq 0$, then Theorem 3.1 implies that the systems of linear equations with matrices A and B are equivalent. By Theorem 4.1, the system of homogeneous linear equations with matrix A has a nonzero solution. This means that the system of homogeneous linear equations with matrix B also has a nonzero solution, and the fact that B is singular follows from Theorem 4.1. ■

Theorem 4.4. *If the matrices*

$$A' = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1n} \\ a'_{i1} & a'_{i2} & \cdots & a'_{in} \\ a_{i+11} & a_{i+12} & \cdots & a_{i+1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix} \quad \textit{and}$$

$$A'' = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i-1\,1} & a_{i-1\,2} & \dots & a_{i-1\,n} \\ a''_{i1} & a''_{i2} & \dots & a''_{in} \\ a_{i+1\,1} & a_{i+1\,2} & \dots & a_{i+1\,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}$$

are singular, the matrix

$$A = \begin{Vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i-1\,1} & a_{i-1\,2} & \dots & a_{i-1\,n} \\ a'_{i1}+a''_{i1} & a'_{i2}+a''_{i2} & \dots & a'_{in}+a''_{in} \\ a_{i+1\,1} & a_{i+1\,2} & \dots & a_{i+1\,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{Vmatrix}$$

is singular, too.

▶ By Theorem 4.1, the systems of homogeneous linear equations with matrices A' and A'' have nonzero solutions. Suppose that

$$\bar{u}' = (\alpha'_1, \dots, \alpha'_n) \quad \text{and} \quad \bar{u}'' = (\alpha''_1, \dots, \alpha''_n)$$

are these solutions. It is clear that both $\bar{u}'$ and $\bar{u}''$ satisfy all the equations in the systems of homogeneous linear equations with matrix A with the exception, perhaps, of the ith equation in A. By Theorem 3.5, any rows of the form $\lambda\bar{u}' - \mu\bar{u}''$, where λ and μ are arbitrary real numbers, satisfy these equations, too. Suppose that

$$\lambda = a'_{i1}\alpha''_1 + \dots + a'_{in}\alpha''_n \quad \text{and} \quad \mu = a''_{i1}\alpha'_1 + \dots + a''_{in}\alpha'_n.$$

Then

$$\begin{aligned} &(a'_{i1} + a''_{i1})(\lambda\alpha'_1 - \mu\alpha''_1) \\ &\qquad + \dots + (a'_{in} + a''_{in})(\lambda\alpha'_n - \mu\alpha''_n) \\ &\qquad = \lambda(a'_{i1}\alpha'_1 + \dots + a'_{in}\alpha'_n) \\ &\qquad + \lambda(a''_{i1}\alpha'_1 + \dots + a''_{in}\alpha'_n) \\ &\qquad - \mu(a'_{i1}\alpha''_1 + \dots + a'_{in}\alpha''_n) \\ &\qquad - \mu(a''_{i1}\alpha''_1 + \dots + a''_{in}\alpha''_n) \\ &\qquad = \lambda 0 + \lambda\mu - \mu\lambda - \mu 0 = 0. \end{aligned}$$

Hence, if $\lambda\bar{u}' - \mu\bar{u}'' \neq \bar{0}$, it proves to be a nonzero solution to the system of homogeneous linear equations with matrix A, which in view of Theorem 4.1 proves the singularity of A. But suppose that $\lambda\bar{u}' - \mu\bar{u}'' = \bar{0}$. If, in addition, $\mu \neq 0$, then $\bar{u}'' = (\lambda/\mu)\,\bar{u}'$, whence

$$(a'_{i1} + a''_{i1})\,\alpha''_1 + \ldots + (a'_{in} + a''_{in})\,\alpha''_n$$
$$= \frac{\lambda}{\mu}\,(a'_{i1}\alpha'_1 + \ldots + a'_{in}\alpha'_n)$$
$$+ (a''_{i1}\alpha''_1 + \ldots + a''_{in}\alpha''_n)$$
$$= \frac{\lambda}{\mu} 0 + 0 = 0.$$

This means that the system of homogeneous linear equations with matrix A again has a nonzero solution, $\bar{u}''$, and, just as in the previous case, the singularity of A follows from Theorem 4.1. If, finally, $\mu = 0$, then

$$(a'_{i1} + a''_{i1})\,\alpha'_1 + \ldots + (a'_{in} + a''_{in})\,\alpha''_n$$
$$= (a'_{i1}\alpha'_1 + \ldots + a'_{in}\alpha'_n)$$
$$+ (a''_{i1}\alpha'_1 + \ldots + a''_{in}\alpha'_n)$$
$$= 0 + \mu = 0 + 0 = 0.$$

Again the system of homogeneous equations with matrix A proves to have a nonzero solution $\bar{u}'$, and we can once more employ Theorem 4.1. ■

If in a matrix A some k rows and k columns are isolated, the elements at the intersections of these rows and columns form a square $k \times k$ matrix, a *submatrix* of A. The highest order of the nonsingular submatrices of A is said to be the *rank* of matrix A. Clearly, the rank of an $m \times n$ matrix cannot exceed the smallest of the numbers m and n. If a matrix has no nonsingular submatrices, its rank is zero by definition. It is quite obvious that null matrices are the only matrices that contain no nonsingular submatrices, which implies that the rank of a matrix is zero if and only if this is a null matrix.

Theorem 4.5. *The rank of a step-like matrix is equal to the number of the nonzero rows in the matrix.*

▶ If $A = 0$, the rank of A is zero by definition. But suppose that A is a nonzero step-like matrix with r nonzero rows. Then, specifying the nonzero rows and columns that contain the leaders of the above-mentioned rows, we arrive at a step-like submatrix that contains no zero rows. By Theorem 4.1, this submatrix is nonsingular, so that A contains a nonsingular submatrix of order r. Every submatrix of A of higher order contains a zero row and is therefore singular (by definition). ■

Highly important for our further investigation is the following

Theorem 4.6. *The rank of a matrix does not change under elementary transformations of the matrix rows.*

First let us prove the following

Lemma. *If going over from a matrix A to a matrix B requires a finite number of elementary transformations of rows, then* (rank B) $\leqslant$ (rank A).

We will establish the validity of this lemma for the case where there is only one elementary transformation. Let us put (rank A) $= r$. To prove the lemma it is sufficient to make sure that every submatrix M of B with an order greater than r is singular. If going over from A to B requires interchanging two rows, submatrix M either coincides with a submatrix M' of A whose order is greater than r or differs from such a submatrix M' in the position of the rows. Since (rank A) $= r$, we conclude that M' is a singular matrix, and the singularity of M follows from Theorem 4.2. Now suppose that going over from A to B requires adding the product of the jth row of A by λ to the ith row of A. Three cases are possible here: (1) the ith row lies outside submatrix M, (2) both the ith row and the jth row pass through submatrix M, and (3) the ith row passes through M while the jth row does not. In the first case submatrix M coincides with the appropriate submatrix of A and, hence, is singular. In the second case we have

$$M = \begin{Vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{ik_1} + \lambda a_{jk_1} & \ldots & a_{ik_s} + \lambda a_{jk_s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{jk_1} & \ldots & a_{jk_s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}.$$

Clearly, matrix M is obtained from matrix

$$M' = \left\| \begin{matrix} \cdot & \cdot & \cdot \\ a_{ik_1} & \ldots & a_{ik_s} \\ \cdot & \cdot & \cdot \\ a_{jk_1} & \ldots & a_{jk_s} \\ \cdot & \cdot & \cdot \end{matrix} \right\|$$

through an elementary transformation of the second kind. But since M' is a submatrix of A with an order greater than r, it is singular. By Theorem 4.2, M must be singular, too. In the third case we write

$$M = \left\| \begin{matrix} \cdot & \cdot & \cdot \\ a_{ik_1} + \lambda a_{jk_1} & \ldots & a_{ik_s} + \lambda a_{jk_s} \\ \cdot & \cdot & \cdot \end{matrix} \right\|,$$

$$M' = \left\| \begin{matrix} \cdot & \cdot & \cdot \\ a_{ik_1} & \ldots & a_{ik_s} \\ \cdot & \cdot & \cdot \end{matrix} \right\| \qquad M'' = \left\| \begin{matrix} \cdot & \cdot & \cdot \\ \lambda a_{jk_1} & \ldots & \lambda a_{jk_s} \\ \cdot & \cdot & \cdot \end{matrix} \right\|,$$

and

$$M''' = \left\| \begin{matrix} \cdot & \cdot & \cdot \\ a_{jk_1} & \ldots & a_{jk_s} \\ \cdot & \cdot & \cdot \end{matrix} \right\|.$$

Matrix M' is a submatrix of A, and M''' differs from a submatrix of A by the arrangement of the rows. Since the orders of both matrices, M' and M''', exceed r, by Theorem 4.2 they are singular. Theorem 4.3 implies the singularity of M'', after which the singularity of M follows from Theorem 4.4. Thus, we have proved the lemma for the case where only one elementary transformation is employed. Now suppose that k elementary transformations are required. Let A, C_1, . . ., C_{k-1}, B be the sequence of matrices that emerges as a result of these transformations. By virtue of what we have just proved,

$$(\text{rank } B) \leqslant (\text{rank } C_{k-1}) \leqslant \ldots \leqslant (\text{rank } C_1) \leqslant (\text{rank } A).$$

▶ Suppose that going over from A to B requires a finite number of elementary transformations. According to the lemma, $(\text{rank } B) \leqslant (\text{rank } A)$. But if these elementary transformations make it possible to go from A to B, then, according to Theorem 2.4, going over from B to A also requires a finite number of elementary transformations. Applying the lemma once more, we find that

(rank A) $\leqslant$ (rank B). The inequalities yield the sought equality. ■

Theorems 2.6, 4.6, and 4.5 provide a practical method for calculating the rank of a matrix: reduce the matrix to step-like form and count the number of nonzero rows in the step matrix. For example, the calculations carried out at the end of Section 2 show that

$$\operatorname{rank}\begin{Vmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & 1 & 0 & 2 & 4 \\ 1 & 1 & -1 & 3 & 4 \end{Vmatrix} = 2,$$

$$\operatorname{rank}\begin{Vmatrix} 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{Vmatrix} = \operatorname{rank}\begin{Vmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 4 & 9 & 14 \end{Vmatrix} = 3.$$

Exercises

1. Find the rank of the matrices of Exercise 7 of Section 2.
2. Find the ranks of the following matrices:

(a) $\begin{Vmatrix} 18 & 7 & 4 & 8 \\ 10 & 1 & 0 & 4 \\ 17 & 3 & 1 & 7 \\ 40 & 17 & 10 & 18 \end{Vmatrix}$;

(b) $\begin{Vmatrix} 1 & -1 & 2 & 1 & -1 & -1 \\ -1 & 2 & -7 & -5 & 6 & 0 \\ 8 & -4 & 3 & -1 & -2 & -5 \\ 1 & 0 & -2 & -2 & 2 & -1 \\ 3 & -1 & -1 & -2 & 1 & -2 \end{Vmatrix}$;

(c) $\begin{Vmatrix} 1 & 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 & -1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \\ 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 & -1 \end{Vmatrix}$;

(d) $\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{Vmatrix}$;

(e) $\begin{Vmatrix} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \dots & a_2+b_n \\ \dots & \dots & \dots & \dots \\ a_n+b_1 & a_n+b_2 & \dots & a_n+b_n \end{Vmatrix}$.

3. Prove that if A and B are matrices with the same number of columns, then

$$\left(\text{rank}\left\|\begin{array}{|c|}\hline A\\ \hline B\\ \hline\end{array}\right\|\right)\leqslant(\text{rank } A)+(\text{rank } B).$$

4. Prove that if A and B are matrices with the same number of rows, then

$$\left(\text{rank}\left\|\begin{array}{|c|c|}\hline A & B\\ \hline\end{array}\right\|\right)\leqslant(\text{rank } A)+(\text{rank } B).$$

5. Prove that if A and B are matrices with the same number of rows and the same number of columns, then

$$\text{rank}\begin{Vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n}\\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n}\\ \dots & \dots & \dots & \dots\\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn}\end{Vmatrix}\leqslant$$

$$\leqslant(\text{rank } A)+(\text{rank } B).$$

Hint. Use the result of Exercise 3.

6. Suppose that A is a nonsingular $n\times n$ matrix, B is a $p\times q$ matrix, C is an $n\times q$ matrix, and O is a null $p\times n$ matrix. Prove that

(a)
$$\text{rank}\begin{Vmatrix} A & C\\ O & B\end{Vmatrix}=n+(\text{rank } B),\quad\text{and}$$

(b) if kD is the matrix obtained by multiplying all the matrix elements of a matrix D by a number k and if $p = n$, then

$$\text{rank}\begin{Vmatrix} A & B\\ 2A & 5B\end{Vmatrix}=n+(\text{rank } B).$$

7. Prove that the rank of a matrix does not change if rows that are linear combinations of the matrix rows are adjoined to the matrix.

8. Prove that by applying elementary transformations of all three kinds to rows and columns we can transform any matrix of rank r into a matrix D such that $d_{11} = \dots = d_{rr} = 1$ while all the other matrix elements are zeros.

9. Prove that if A is a (square) $n\times n$ matrix whose rank is n, then by applying elementary transformations of the second kind to the rows and columns we can reduce A to the form

$$\begin{Vmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & 0 & d\end{Vmatrix}.$$

10. Prove that if a column in a square matrix A is a linear combination of the other columns, A is singular.

Hint. Use Theorem 4.1.

5. THE THEOREM ON PRINCIPAL UNKNOWNS

A system of linear equations is said to be *consistent* if it has a solution. In particular, a system of homogeneous linear equations is always consistent.

Theorem 5.1. (Kronecker-Capelli theorem). *A system of linear equations is consistent if and only if the rank of the system matrix is equal to the rank of the augmented matrix.*

▶ Since according to Theorems 2.6 and 3.2 from each system of linear equations we can go over to an equivalent step-like system and according to Theorem 4.6 the ranks of the system matrix and augmented matrix do not change under such transformations, it is sufficient to prove the theorem for step-like systems. But for a step-like system, according to Theorem 4.5, the ranks of the system matrix and the augmented matrix are the same if and only if these matrices have an equal number of nonzero rows, or, which is the same thing, if and only if the first nonzero element in the last nonzero row of the augmented matrix is not in the column consisting of the absolute terms. From the analysis of the step-like system carried out in Section 3 the reader knows that this happens if and only if the system is consistent. ■

When in Section 3 we spoke of principal and absolute unknowns, we found that these concepts depend on the way the augmented matrix is reduced to step-like form. To solve the problems we discussed at the end of Section 3, we need a definition that depends only on the given system of linear equations. With this in mind, let us proceed as follows. If we have a system of linear equations in the unknowns $x_1, x_2, \ldots, x_n$, we will say that the *unknowns* $x_{i_1}, \ldots, x_{i_k}$ *can be declared principal* if the values of these unknowns are determined uniquely whatever the values of the other unknowns. As for the other unknowns, we will say that they *can be declared absolute.* Note that here we are defining the notions "can be declared principal" and "can be declared absolute" rather than the notions "principal unknowns" and "absolute unknowns". The unknowns become principal or absolute only after we declare them such in the process of realization of the available possibility. It is clear that the principal and absolute unknowns defined in the sense

of the definition in Section 3 can be declared such in the sense of our new definition.

The central result of our theory is

Theorem 5.2 (theorem on principal unknowns). *Suppose we have a consistent system of m linear equations in n unknowns, $\tilde{A}$ is the augmented matrix of this system, and* (rank $\tilde{A}$) $= r$. *Then the unknowns $x_{i_1}, \ldots, x_{i_k}$ can be declared principal if and only if $k = r$ and the columns of matrix $\tilde{A}$ with numbers $i_1, \ldots, i_k$ contain the elements of a nonsingular submatrix of order r.*

▶ Suppose that $k = r$ and there is a nonsingular submatrix of order r in the columns with the above-stated numbers. Then the rank of the $m \times n$ matrix

$$\begin{Vmatrix} a_{1i_1} & a_{1i_2} & \ldots & a_{1i_r} \\ a_{2i_1} & a_{2i_2} & \ldots & a_{2i_r} \\ \cdot & \cdot & \cdot & \cdot \\ a_{mi_1} & a_{mi_2} & \ldots & a_{mi_r} \end{Vmatrix}$$

is r. Reducing this matrix to step-like form, we arrive, via Theorems 4.5 and 4.6, at the following matrix:

$$\begin{Vmatrix} b_{1i_1} & b_{1i_2} & \ldots & b_{1i_r} \\ 0 & b_{2i_2} & \ldots & b_{2i_r} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & b_{ri_r} \\ 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 \end{Vmatrix}$$

with $b_{1i_1}, b_{2i_2}, \ldots, b_{ri_r} \neq 0$. Applying the same elementary transformations to matrix $\tilde{A}$, we get

$$\tilde{B} = \left\Vert \begin{array}{ccccccccc|c} \ldots & b_{1i_1} & \ldots & b_{1i_2} & \ldots & b_{1i_r} & \ldots & c_1 \\ \ldots & 0 & \ldots & b_{2i_2} & \ldots & b_{2i_r} & \ldots & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \ldots & 0 & \ldots & 0 & \ldots & b_{ri_r} & \ldots & c_r \\ \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & c_{r+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & c_m \end{array} \right\Vert .$$

Note that this notation does not mean that all the rows of $\widetilde{B}$ starting with the $(r+1)$st are zero rows: the columns whose numbers differ from $i_1, i_2, \ldots, i_r$ may contain nonzero elements in these rows. Actually, however, all rows of $\widetilde{B}$ starting with the $(r+1)$st are zero rows. Indeed, suppose that this is not so, that is, $b_{pq} = 0$ for some p and q, with $r+1 \leqslant p \leqslant m$. Of course, $q \neq i_1, \ldots, i_r$. But it may so happen that $q = n+1$, that is, $b_{pq} = c_p$. Let M be the submatrix of $\widetilde{B}$ of order $r+1$ whose elements are in the rows with numbers $1, 2, \ldots, r, p$ and in columns with numbers $i_1, \ldots, i_r, q$. Here three cases are possible: (1) $q < i_1$, (2) $i_s < q < i_{s+1}$ for a certain s, and (3) $i_r < q$. Interchanging, when necessary, the rows in matrix M, we arrive at a matrix M' equal to

$$\begin{Vmatrix} b_{pq} & 0 & 0 & \ldots & 0 \\ b_{1q} & b_{1i_1} & b_{1i_2} & \ldots & b_{1i_r} \\ b_{2q} & 0 & b_{2i_2} & \ldots & b_{2i_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{rq} & 0 & 0 & \ldots & b_{ri_r} \end{Vmatrix},$$

$$\begin{Vmatrix} b_{1i_1} & \ldots & b_{1i_s} & b_{1q} & b_{1i_{s+1}} & \ldots & b_{1i_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & b_{si_s} & b_{sq} & b_{si_{s+1}} & \ldots & b_{si_r} \\ 0 & \ldots & 0 & b_{pq} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & b_{s+1q} & b_{s+1i_{s+1}} & \ldots & b_{s+1i_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & 0 & b_{rq} & 0 & \ldots & b_{ri_r} \end{Vmatrix},$$

and

$$\begin{Vmatrix} b_{1i_1} & b_{1i_2} & \ldots & b_{1i_r} & b_{1q} \\ 0 & b_{2i_2} & \ldots & b_{2i_r} & b_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & b_{ri_r} & b_{rq} \\ 0 & 0 & \ldots & 0 & b_{pq} \end{Vmatrix},$$

respectively. Since $b_{1i_1}, b_{2i_2}, \ldots, b_{ri_r}, b_{pq} \neq 0$, in the third case matrix M' proves to be step-like. In the first

two cases matrix M' can be transformed into a step-like matrix via elementary transformations of the second kind by nullifying the elements $b_{1q}, b_{2q}, \ldots, b_{rq}$ and $b_{s+1\,q}, \ldots, b_{rq}$, respectively; see Theorem 2.3. By virtue of Theorems 4.5 and 4.6,

$$(\operatorname{rank} M) = (\operatorname{rank} M') = r + 1.$$

Hence, M is a nonsingular matrix, and, taking into account the definition of the rank of a matrix and Theorem 4.6, we get

$$(\operatorname{rank} \widetilde{A}) = (\operatorname{rank} \widetilde{B}) \geqslant r + 1,$$

which contradicts the hypothesis. Thus, discarding the zero rows in matrix $\widetilde{B}$, we arrive at a system of linear equations consisting of r equations. Assigning arbitrary values to the unknowns that differ from $x_{i_1}, \ldots, x_{i_r}$ and transferring them to the right-hand side, we can easily see that the unknowns $x_{i_1}, \ldots, x_{i_r}$ are uniquely determined one right after the other, starting from the last one. Thus, they can be declared principal. By virtue of Theorems 1.1 and 3.2, the same is true of the initial system of linear equations.

Now let us assume that the unknowns $x_{i_1}, \ldots, x_{i_k}$ can be declared principal. Nullifying the other unknowns, we arrive at the system

$$\begin{array}{c} a_{1i_1}x_{i_1} + \ldots + a_{1i_k}x_{i_k} = b_1, \\ \ldots\ldots\ldots\ldots\ldots \\ a_{mi_1}x_{i_1} + \ldots + a_{mi_k}x_{i_k} = b_m, \end{array} \tag{5.1}$$

which has a unique solution. Suppose that $\widetilde{C}$ is the augmented matrix of (5.1) and $\widetilde{B}$ is the step-like matrix to which matrix $\widetilde{C}$ can be reduced by Theorem 2.6. By virtue of Theorem 3.2, the corresponding system of linear equations has a unique solution. As the analysis of a step-like system carried out in Section 3 has shown, this implies that matrix $\widetilde{B}$ contains k nonzero rows with the leaders in the first, second, third, etc. columns. In other words,

$$\widetilde{B} = \left\| \begin{array}{cccc|c} b_{11} & b_{12} & \ldots & b_{1k} & c_1 \\ 0 & b_{22} & \ldots & b_{2k} & c_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & b_{kk} & c_k \end{array} \right\|.$$

By virtue of Theorems 4.5 and 4.6,

$$k = \operatorname{rank} \widetilde{B} = \operatorname{rank} \widetilde{C} \leqslant \operatorname{rank} \widetilde{A} = r.$$

Let us assume that $k < r$. To matrix $\widetilde{A}$ we apply the elementary transformations that transform $\widetilde{C}$ into $\hat{B}$. Then matrix $\widetilde{A}$ transforms into

$$\widetilde{D} = \left\| \begin{array}{ccccccccc|c} \ldots & b_{11} & \ldots & b_{12} & \ldots & b_{1k} & \ldots & & & c_1 \\ \ldots & 0 & \ldots & b_{22} & \ldots & b_{2k} & \ldots & & & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot\ \cdot\ \cdot \\ \ldots & 0 & \ldots & 0 & \ldots & b_{kk} & \ldots & & & c_k \\ \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & & & c_{k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \cdot\ \cdot\ \cdot \\ \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & & & c_m \end{array} \right\|,$$

where the columns occupied by the columns of $\widetilde{B}$ have the numbers $i_1, \ldots, i_k, n + 1$. By Theorem 4.6 (rank $\widetilde{D}$) $=$ r, while by Theorem 3.2 the system of linear equations with the augmented matrix $\widetilde{D}$ is consistent and the unknowns $x_{i_1}, \ldots, x_{i_k}$ can be declared principal. From Theorem 5.1 it follows that matrix D, which is obtained from $\widetilde{D}$ by discarding the last column, has the same rank as $\widetilde{D}$. Since $k < r$, among the rows of D with numbers greater than k there are nonzero rows, since otherwise D could not contain nonsingular submatrices of order r. However, the nonzero element of such a row cannot be in a column occupied by a column of $\widetilde{B}$. Thus, $\widetilde{D}$ has matrix elements $d_{pq} \neq 0$, with $p > k$ and $q \neq i_1, \ldots, i_k$, $n + 1$. If we now nullify all the unknowns with numbers that differ from $i_1, \ldots, i_k, q$, we get $x_q = c_p/d_{pq}$. Therefore, x_q cannot assume arbitrary values, which contradicts the possibility of declaring $x_{i_1}, \ldots, x_{i_k}$ principal. Hence, $k = r$, which completes the proof of the theorem. ■

It follows from the above proof that in system (1.6) we can declare principal only the pairs x_1 and x_3 or x_2 and x_3 or x_3 and x_4, while other pairs cannot be declared principal.

Let us dwell on another method for solving systems of homogeneous linear equations. From Theorem 3.5 it follows that the set of all the solutions of a given system

of homogeneous linear equations in n unknowns combined with any family of n-dimensional rows contains linear combinations of these solutions. It is therefore natural to try and find a family of solutions (preferably with the smallest number of solutions possible) such that all other solutions prove to be linear combinations of these. One such family is specified by the following

Theorem 5.3. *Suppose that the unknowns $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$ in a system of homogeneous linear equations in n unknowns with matrix A of rank r can be declared principal. For each k, with $1 \leqslant k \leqslant n-r$, by $\bar{u}_k$ we denote the unique solution of the system that is obtained if x_{i_k} is assigned the value of 1 while all the other absolute unknowns are assigned the value of 0. Then every solution to the system considered is a linear combination of the family $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-r}$.*

▶ Suppose that

$$\bar{v} = (v_1, \ldots, v_n)$$

is an arbitrary solution to the system considered. Take the row

$$\bar{w} = v_{i_1}\bar{u}_1 + v_{i_2}\bar{u}_2 + \ldots + v_{i_{n-r}}\bar{u}_{n-r}.$$

According to Theorem 3.5, $\bar{w}$ is also a solution to this system. One can easily see that the i_1st, i_2nd, ..., i_{n-r}th coordinates of row $\bar{w}$ are equal to $v_{i_1}, v_{i_2}, \ldots, v_{i_{n-r}}$, respectively. In other words, the values of the unknowns $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-r}}$ for solution $\bar{v}$ are the same as for solution $\bar{w}$. But from the definition of the possibility of declaring some unknowns principal and some absolute it follows that fixing the values of the above unknowns uniquely determines the values of the other unknowns. Hence, the remaining coordinates of row $\bar{v}$ must coincide with those of row $\bar{w}$, that is

$$\bar{v} = \bar{w} = v_{i_1}\bar{u}_1 + v_{i_2}\bar{u}_2 + \ldots + v_{i_{n-r}}\bar{u}_{n-r},$$

which is what we set out to prove. ■

The fact that it is impossible to decrease the number of solutions in the family of solutions found above follows from

Theorem 5.4. *If every solution to a system of homogeneous linear equations in n unknowns with matrix A of rank r is*

a linear combination of solutions $\bar{v}_1, \ldots, \bar{v}_s$, then $s \geqslant n - r$.

▶ Suppose that s is smaller than $n - r$. We put

$$\bar{v}_i = (\alpha_{i1}, \ldots, \alpha_{in}) \quad (i = 1, 2, \ldots, s)$$

and, allowing for Theorem 5.2, we denote the solutions considered in Theorem 5.3 by

$$\bar{u}_j = (\beta_{j1}, \ldots, \beta_{jn}) \quad (j = 1, 2, \ldots, n - r).$$

By hypothesis,

$$\bar{u}_j = a_{1j}\bar{v}_1 + \ldots + a_{sj}\bar{v}_s \quad (j = 1, 2, \ldots, n - r),$$

with the appropriate choice of the real numbers a_{ij}. Consider the system of homogeneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1\,n-r}x_{n-r} &= 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2\,n-r}x_{n-r} &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{s\,n-r}x_{n-r} &= 0. \end{aligned}$$

By assumption, the number of equations in this system is smaller than the number of unknowns and, by Theorem 3.3, it has a nonzero solution, say $(\xi_1, \ldots, \xi_{n-r})$. Note that the real numbers $\xi_1, \ldots, \xi_{n-r}$ are the i_1st, i_2nd, ..., i_{n-r}th coordinates of the row

$$\bar{w} = \xi_1\bar{u}_1 + \xi_2\bar{u}_2 + \ldots + \xi_{n-r}\bar{u}_{n-r}.$$

Taking into account Theorems 2.1 and 2.2, we get

$$\begin{aligned} \bar{0} \neq \bar{w} &= \xi_1 (a_{11}\bar{v}_1 + a_{21}\bar{v}_2 + \ldots + a_{s1}\bar{v}_s) \\ &+ \xi_2 (a_{12}\bar{v}_1 + a_{22}\bar{v}_2 + \ldots + a_{s2}\bar{v}_s) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \xi_{n-r} (a_{1\,n-r}\bar{v}_1 + a_{2\,n-r}\bar{v}_2 + \ldots + a_{s\,n-r}\bar{v}_s) \\ &= (\xi_1 a_{11} + \xi_2 a_{12} + \ldots + \xi_{n-r}a_{1\,n-r})\,\bar{v}_1 \\ &+ (\xi_1 a_{21} + \xi_2 a_{22} + \ldots + \xi_{n-r}a_{2\,n-r})\,\bar{v}_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ (\xi_1 a_{s1} + \xi_2 a_{s2} + \ldots + \xi_{n-r}a_{s\,n-r})\,\bar{v}_s \\ &= 0\bar{v}_1 + 0\bar{v}_2 + \ldots + 0\bar{v}_s = \bar{0}. \end{aligned}$$

This contradiction completes the proof of the theorem. ■

Thus, Theorem 5.3 shows that all the solutions to a system of homogeneous linear equations can be obtained by considering the various linear combinations of a certain finite system of equations.

To obtain a description as graphic as this one for the set of all solutions to a system of inhomogeneous linear equations, we turn the reader's attention to the following facts:

Theorem 5.5 (a) *If $\bar{u}$ is a solution to a system of linear equations with matrix A and $\bar{v}$ is a solution to the system of homogeneous equations with the same matrix A, then $\bar{u} + \bar{v}$ is a solution to the first system.*

(b) *If $\bar{u}_0$ is a solution to a system of linear equations with matrix A, then every solution $\bar{u}$ to this system can be written in the form of a sum $\bar{u}_0 + \bar{v}$, where $\bar{v}$ is a solution to the system of homogeneous linear equations with the same matrix A.*

▶ Suppose that b_i is the absolute term in the ith equation in the system considered.

(a) If $\bar{u} = (\alpha_1, \ldots, \alpha_n)$ and $\bar{v} = (\beta_1, \ldots, \beta_n)$, then, substituting the coordinates of the row $\bar{u} + \bar{v}$ into the ith equation of the system considered, we obtain

$$\begin{aligned} a_{i1}(\alpha_1 + \beta_1) + \ldots + a_{in}(\alpha_n + \beta_n) &= (a_{i1}\alpha_1 + \ldots + a_{in}\alpha_n) \\ &\quad + (a_{i1}\beta_1 + \ldots + a_{in}\beta_n) \\ &= b_i + 0 = b_i, \end{aligned}$$

which is what we set out to prove.

(b) If $\bar{u}_0 = (\gamma_1, \ldots, \gamma_n)$ and $\bar{u} = (\alpha_1, \ldots, \alpha_n)$, then

$$\begin{aligned} a_{i1}(\alpha_1 - \gamma_1) + \ldots + a_{in}(\alpha_n - \gamma_n) &= (a_{i1}\alpha_1 + \ldots + a_{in}\alpha_n) \\ &\quad - (a_{i1}\gamma_1 + \ldots + a_{in}\gamma_n) \\ &= b_i - b_i = 0. \end{aligned}$$

Hence, the row $\bar{v} = \bar{u} - \bar{u}_0$ is a solution to the system of homogeneous equations with matrix A, which means that the equation

$$\bar{u} = \bar{u}_0 + (\bar{u} - \bar{u}_0) = \bar{u}_0 + \bar{v}$$

provides the sought representation. ■

From Theorem 5.5 it follows that to describe the set of all the solutions to an arbitrary system of linear equations with matrix A one must find a solution to this system and then add to this solution arbitrary linear combinations of the solutions (described in Theorem 5.3) to the system of homogeneous equations with the same matrix A. By way of an example, let us consider the system

$$x_1 + x_2 + x_3 + x_4 = 4,$$

$$x_1 + x_2 - x_3 + x_4 = 2.$$

It is clear that the row (1, 1, 1, 1) is a solution to this system. For this reason, an arbitrary solution has the form

$$(1, 1, 1, 1) + \alpha (1, 0, 0, -1) + \beta (0, 1, 0, -1)$$

or, which is the same,

$$(1 + \alpha, 1 + \beta, 1, 1 - \alpha - \beta),$$

where α and β are arbitrary real numbers.

Exercises

1. Which unknowns in the system of linear equations of Exercises 1 and 2 in Section 3 can be declared principal?

2. Prove that the consistent system of linear equations in n unknowns has a unique solution if and only if the rank of the system matrix is equal to n.

3. Prove that if a system of n linear equations in $n - 1$ unknowns is consistent, the augmented matrix of this system is singular.

4. Find the set of solutions described in Theorem 5.3 for the systems of linear equations of Exercises 1(h)-1(k) in Section 3.

5. Find the set of solutions described in Theorem 5.3 for the system of linear equations that solves Exercise 1(f) in Section 1.

6. Describe the solutions to the systems of linear equations of Exercises 1(a)-1(e) in Section 3 by the method developed at the end of this section.

7. The same as in Exercise 7 but for the systems of linear equations that solve Exercises 1(a), 1(b), and 1(e) in Section 1.

8. Suppose that $\bar{u}_1, \ldots, \bar{u}_r$ are solutions to a system of inhomogeneous linear equations. Prove that the row $\lambda_1\bar{u}_1 + \ldots + \lambda_r\bar{u}_r$ is the solution to the system if and only if $\lambda_1 + \ldots + \lambda_r = 1$.

6. FUNDAMENTAL SYSTEMS OF SOLUTIONS

As we have said earlier, Theorem 5.3 shows that all the solutions to a system of homogeneous equations can be obtained by constructing linear combinations of a finite set of solutions. The natural question is: how is one to know whether a given set of solutions provides the means for this? The aim of this section is to give an answer to this question.

The *rank of a system* $\{\bar{u}_1, \ldots, \bar{u}_m\}$ of rows of length n is the rank of the $m \times n$ matrix constructed from the rows belonging to the system. For example, the rank of the system of rows

$$\{(1, 1, 1, 1, 4), (1, 1, 2, 0, 4), (1, 1, 0, 2, 4), (1, 1, -1, 3, 4)\}$$

is equal to the rank of the matrix

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 2 & 0 & 4 \\ 1 & 1 & 0 & 2 & 4 \\ 1 & 1 & -1 & 3 & 4 \end{Vmatrix},$$

that, as noted on p. 37, is equal to 2.

Let us establish some additional facts. The reader will recall that a row $\bar{v}$ is, by definition, a *linear combination* of rows $\bar{u}_1, \ldots, \bar{u}_m$ if $\bar{v} = \lambda_1\bar{u}_1 + \ldots + \lambda_m\bar{u}_m$ for a set of real numbers $\lambda_1, \ldots, \lambda_m$. In this case it is often said that $\bar{v}$ is *expressed linearly* in terms of rows $\bar{u}_1, \ldots, \bar{u}_m$. For example, the row $\bar{w} = (3, 1, 2, 2)$ can be expressed linearly in terms of the rows $\bar{u} = (1, 2, 1, 2)$ and $\bar{v} = (-1, 3, 0, 2)$, since $\bar{w} = 2\bar{u} + (-1)\bar{v}$.

Theorem 6.1. *If a row $\bar{w}$ can be expressed linearly in terms of rows $\bar{u}_1, \ldots, \bar{u}_s$ and each of the $\bar{u}_i$ can be expressed linearly in terms of rows $\bar{v}_1, \ldots, \bar{v}_t$, then $\bar{w}$ can be expressed linearly in terms of $\bar{v}_1, \ldots, \bar{v}_t$.*

▶ By hypothesis,

$$\bar{w} = \lambda_1\bar{u}_1 + \ldots + \lambda_s\bar{u}_s \quad \text{and} \quad \bar{u}_i = \mu_{i1}\bar{v}_1 + \ldots + \mu_{it}\bar{v}_t$$

for appropriate real numbers $\lambda_1, \ldots, \lambda_s, \mu_{i1}, \ldots, \mu_{it}$. Whence, taking into account Theorems 2.1 and 2.2,

we get

$$
\begin{aligned}
\bar{w} &= \lambda_1 (\mu_{11}\bar{v}_1 + \mu_{12}\bar{v}_2 + \dots + \mu_{1t}\bar{v}_t) \\
&+ \lambda_2 (\mu_{21}\bar{v}_1 + \mu_{22}\bar{v}_2 + \dots + \mu_{2t}\bar{v}_t) \\
&\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
&+ \lambda_s (\mu_{s1}\bar{v}_1 + \mu_{s2}\bar{v}_2 + \dots + \mu_{st}\bar{v}_t) \\
&= (\lambda_1\mu_{11} + \lambda_2\mu_{21} + \dots + \lambda_s\mu_{s1})\, \bar{v}_1 \\
&+ (\lambda_1\mu_{12} + \lambda_2\mu_{22} + \dots + \lambda_s\mu_{s2})\, \bar{v}_2 \\
&\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\
&+ (\lambda_1\mu_{1t} + \lambda_2\mu_{2t} + \dots + \lambda_s\mu_{st})\, \bar{v}_t,
\end{aligned}
$$

which is what we set out to prove. ■

Theorem 6.2. *Suppose that*

$$
W = \begin{Vmatrix} u_{11} & \dots & u_{1n} \\ \cdot & \cdot & \cdot \\ u_{s1} & \dots & u_{sn} \\ v_{11} & \dots & v_{1n} \\ \cdot & \cdot & \cdot \\ v_{t1} & \dots & v_{tn} \end{Vmatrix}.
$$

Then (rank U) $\leqslant$ (rank W). *But if the rows of matrix V can be expressed linearly in terms of the rows of matrix U, then* (rank W) = (rank U).

▶ To prove the first assertion it is sufficient to note that every nonsingular submatrix of matrix U is also a nonsingular submatrix of matrix W. Hence, the highest order of such submatrices of matrix U cannot exceed that of submatrices of matrix W. By virtue of the definition of the rank of a matrix, this means that (rank U) $\leqslant$ (rank W). Going over to the proof of the second assertion, we put $\bar{u}_i = (u_{i1}, \dots, u_{in})$ and $\bar{v}_j = (v_{j1}, \dots, v_{jn})$. By hypothesis

$$
\bar{v}_j = \lambda_{j1}\bar{u}_1 + \dots + \lambda_{js}\bar{u}_s
$$

for appropriate real numbers $\lambda_{j1}, \dots, \lambda_{js}$. Subtracting from the $(s + 1)$st row of W the first s rows multiplied by $\lambda_{11}, \lambda_{12}, \dots, \lambda_{1s}$, respectively, we reduce the $(s + 1)$st row to a zero row. Acting in the same manner on the

subsequent rows of W, we arrive at the matrix

$$T = \begin{Vmatrix} u_{11} & \dots & u_{1n} \\ \cdot & \cdot & \cdot \\ u_{s1} & \dots & u_{sn} \\ 0 & \dots & 0 \\ \cdot & \cdot & \cdot \\ 0 & \dots & 0 \end{Vmatrix}.$$

By Theorem 4.6, (rank T) = (rank W). Reducing matrix U to step-like form S, which is possible by virtue of Theorem 2.6, we note that the same elementary transformations transform matrix T into the step-like matrix $\begin{Vmatrix} S \\ O \end{Vmatrix}$, where O is a $r \times n$ null matrix. By virtue of Theorems 4.5 and 4.6, (rank W) = (rank T) = rank $\begin{Vmatrix} S \\ O \end{Vmatrix}$ = (rank S) = (rank U). ■

Theorem 6.3. *If a finite number of elementary transformations of rows is needed to go over from a matrix U to a matrix V, then each of the rows of matrix V can be linearly expressed in terms of the rows of matrix U.*

▶ When only one elementary transformation is required to go over from U to V, the validity of the theorem follows from the definition. In the general case we have a sequence of matrices U, W_1, W_2, . . ., W_k, V, where going over from a left matrix to a right matrix requires only one elementary transformation. Hence, as noted earlier, each row of a particular right matrix can be expressed linearly in terms of the rows of the adjacent left matrix. We need only apply Theorem 6.1 several times. ■

Theorem 6.4. *If each row of a system Ω can be expressed linearly in terms of the rows of another system Ξ then* (rank Ω) $\leqslant$ (rank Ξ).

▶ If U and V are matrices constructed from the rows of systems Ξ and Ω, respectively, then, using Theorem 6.2, we obtain

$$(\text{rank } \Omega) = (\text{rank } V) \leqslant \left(\text{rank} \begin{Vmatrix} U \\ V \end{Vmatrix}\right) = (\text{rank } U)$$
$$= (\text{rank } \Xi). \ ■$$

Theorem 6.5. *If $\Xi = \{\bar{u}_1, \dots, \bar{u}_m\}$ is a system of rows of length n and* (rank Ξ) *is lower than m, then at least*

one of the rows in Ξ can be expressed linearly in terms of the other rows.

▶ Suppose that $\bar{u}_i = (u_{i1}, \ldots, u_{in})$ and U is the matrix constructed from the rows of system Ξ. By hypothesis, $(\text{rank } U) = r < m$. Suppose that $i_1, \ldots, i_r$ are the numbers of rows containing a nonsingular submatrix of U of order r, and V is the $r \times n$ matrix constructed from these rows. Employing Theorem 2.6, we reduce matrix V to step-like form S. Applying the same elementary transformations to matrix U and moving the rows which contain the V matrix to the first r places, we obtain the matrix $W' = \left\| \begin{matrix} S \\ T' \end{matrix} \right\|$, where T' is the $(m - r) \times n$ matrix constructed from the rows of matrix U not included in matrix V. By virtue of Theorems 4.5 and 4.6, all the rows of the step-like matrix S are nonzero. Let us assume that the leaders of the rows of S are in the columns with numbers $j_1, \ldots, j_r$, that is,

$$W' = \left\| \begin{matrix} \ldots & s_{1j_1} & & s_{1j_2} & & s_{1j_r} & \ldots \\ \ldots & 0 & \ldots & s_{2j_2} & & s_{2j_r} & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots & 0 & \ldots & 0 & \ldots & s_{rj_r} & \ldots \\ \ldots & t'_{1j_1} & & t'_{2j_2} & \ldots & t'_{rj_r} & \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \ldots & t'_{n-rj_1} & \ldots & t'_{n-rj_2} & \ldots & t'_{n-rj_r} & \ldots \end{matrix} \right\|,$$

with $s_{1j_1}, s_{2j_2}, \ldots, s_{rj_r} \neq 0$. Applying Theorem 2.3 several times, we can transform matrix W', via elementary transformations, into matrix $W = \left\| \begin{matrix} S \\ T \end{matrix} \right\|$, where all the matrix elements of T that are in the columns with numbers $j_1, \ldots, j_r$ are zeros. Let us make sure that all the rows of T are zero rows. Indeed, suppose that $t_{pq} \neq 0$ for some p and q. Of course, $q \neq j_1, \ldots, j_r$. Take the submatrix M of W whose elements are in the rows with numbers $1, 2, \ldots, r, p$ and in the columns with numbers $j_1, \ldots, j_r, q$. Three cases are possible here: (1) $q < j_1$, (2) $j_h < q < j_{n+1}$ for a certain h, and (3) $j_r < q$. Interchanging, where it is necessary, the rows of M, we arrive

at matrix M equal to

$$\left\|\begin{array}{ccccc} t_{pq} & 0 & 0 & \dots & 0 \\ s_{1q} & s_{1j_1} & s_{1j_2} & \dots & s_{1j_r} \\ s_{2q} & 0 & s_{2j_2} & \dots & s_{2j_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{rq} & 0 & 0 & \dots & s_{rj_r} \end{array}\right\|,$$

$$\left\|\begin{array}{ccccccc} s_{1j_1} & \dots & s_{1j_h} & s_{1q} & s_{1j_{h+1}} & \dots & s_{1j_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & s_{hj_n} & s_{hq} & s_{hj_{h+1}} & \dots & s_{hj_r} \\ 0 & \dots & 0 & t_{pq} & 0 & \dots & 0 \\ 0 & \dots & 0 & s_{h+1q} & s_{h+1j_{h+1}} & \dots & s_{h+1j_r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & s_{rq} & 0 & \dots & s_{rj_r} \end{array}\right\|,$$

or

$$\left\|\begin{array}{ccccc} s_{1j_1} & s_{1j_2} & \dots & s_{1j_r} & s_{1q} \\ 0 & s_{2j_2} & \dots & s_{2j_r} & s_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & s_{rj_r} & s_{rq} \\ 0 & 0 & \dots & 0 & t_{pq} \end{array}\right\|,$$

respectively. Since $s_{1j_1}, s_{2j_2}, \dots, s_{rj_r}, t_{pq} \neq 0$, in the third case matrix M' proves to be step-like at once. In the first two cases, M' can be made step-like via elementary transformations of the second kind, nullifying the elements $s_{1q}, s_{2q}, \dots, s_{rq}$ and $s_{h+1q}, \dots, s_{rq}$, respectively; see Theorem 2.3. By Theorems 4.5, 4.6, and 6.2,

$$r + 1 = (\text{rank } M') = (\text{rank } M) \leqslant (\text{rank } W) = (\text{rank } W') = (\text{rank } U) = r.$$

This contradiction shows that $W = \left\|\begin{array}{c} S \\ O \end{array}\right\|$, where O is an $(m - r) \times n$ null matrix. By virtue of Theorems 2.4 and 6.3, the rows of matrix S can be expressed linearly in terms of the rows of matrix V, while the rows of matrix U can be expressed linearly in terms of the rows of matrix W or, which is the same, in terms of the rows

of matrix S. Whence, employing Theorem 6.1, we conclude that the rows of U that are not included in V can be expressed linearly in terms of the rows of matrix V, which completes the proof of the theorem. ■

The family F of solutions of a system of homogeneous linear equations is said to be the *fundamental system of solutions* (or *fundamental set of solutions*) if every solution to this system of equations is a linear combination of the solutions belonging to F and if discarding at least one solution from F results in the set of the remaining solutions losing this property. By Theorem 5.4, the family of solutions considered in Theorem 5.3 is a fundamental system of solutions. Other fundamental systems of solutions also exist.

Let us consider, by way of an example, the system

$$x_1 + x_2 + x_3 + x_4 = 0,$$
$$x_1 + x_2 - x_3 + x_4 = 0.$$

According to Theorem 5.2, the unknowns x_1 and x_4 can be declared absolute, whereby from Theorem 5.3 it follows that the solutions $(1, -1, 0, 0)$ and $(0, -1, 0, 1)$ form a fundamental system of solutions. But the family $\{(1, 0, 0, -1), (0, 1, 0, -1)\}$ is also a fundamental system of solutions. It emerges if the unknowns x_1 and x_2 are declared absolute. If we take the first fundamental system, we find that the set of all solutions to system (5.2) consists of rows of the form

$$(\alpha, -(\alpha + \beta), 0, \beta),$$

where α and β are arbitrary real numbers. The second fundamental system yields the set of rows of the form

$$(\alpha, \beta, 0, -(\alpha + \beta)).$$

It is easy to see that actually the set of rows has not changed, which was to be expected.

Theorem 6.6. *Each fundamental system of solutions to a system of homogeneous linear equations in n unknowns with a system matrix of rank r contains $n - r$ solutions, and the rank of the fundamental system is $n - r$.*

▶ According to Theorem 5.2, $n - r$ unknowns of the system of linear equations can be declared absolute, which means that there exists a system of solutions $\Xi = \{\bar{u}_1, \ldots, \bar{u}_{n-r}\}$ described in Theorem 5.3. Suppose

that U is the $(n - r) \times n$ matrix constructed from these solutions. The columns of U with numbers $i_1, \ldots, i_{n-r}$ constitute a submatrix E that is the identity matrix of order $n - r$. By Theorems 4.5 and 6.2,

$$n - r = (\text{rank } E) \leqslant (\text{rank } U) \leqslant n - r,$$

whence

$$(\text{rank } \Xi) = (\text{rank } U) = n - r.$$

Now let us assume that F is the fundamental system of solutions to the given system of linear equations and s is the number of solutions in F. By Theorem 5.4, $s \geqslant n - r$. If $s > n - r$, then, by Theorems 6.1 and 6.5, all the solutions to our system can be expressed linearly in terms of a fraction of solutions in F, which contradicts the very definition of F. Finally, applying Theorem 6.4 twice, we get

$$(\text{rank } \Xi) \leqslant (\text{rank } F) \leqslant (\text{rank } \Xi),$$

whence, by virtue of what we have proved above,

$$(\text{rank } F) = (\text{rank } \Xi) = n - r. \blacksquare$$

The problem formulated at the beginning of this section is solved via the following

Theorem 6.7. *If F is a system of solutions to a system of linear equations in n unknowns with a system matrix of rank r, contains $n - r$ solutions, and the rank of F is $n - r$, then F is a fundamental system of solutions.*

▶ Suppose that there exists a solution $\bar{v}$ that is not a linear combination of the solutions belonging to F. Consider a system of solutions, $\bar{F}$, that is obtained by adjoining $\bar{v}$ to F. Since from Theorems 5.3 and 6.6 it follows that the system of linear equations we are considering here has a fundamental system of solutions of rank $n - r$, we conclude, allowing for Theorems 6.2 and 6.4, that $(\text{rank } \bar{F}) = n - r$. But then from Theorem 6.5 and the special choice of solution $\bar{v}$ there follows the existence of a solution $\bar{u} \in F$ such that

$$\bar{u} = \lambda_1 \bar{u}_1 + \ldots + \lambda_s \bar{u}_s + \mu \bar{v},$$

where $\bar{u} \neq \bar{u}_1, \ldots, \bar{u}_s \in F$. If $\mu = 0$, then, by Theorem 6.4, $(\text{rank } F) < n - r$, which contradicts the hypothesis. But if $\mu \neq 0$, then, allowing for Theorems 2.1

and 2.2, we obtain

$$\bar{v} = \mu^{-1}\bar{u} - \mu^{-1}\lambda_1\bar{u}_1 - \ldots - \mu^{-1}\lambda_s\bar{u}_s,$$

which contradicts the choice of $\bar{v}$. Thus, any solution to the system of linear equations can be expressed linearly in terms of the solutions belonging to F. If the same result can be achieved by using only a fraction of solutions belonging to F, then, allowing for Theorems 6.1 and 6.4, we arrive at an impossible relation

$$n - r = (\text{rank } F) < n - r,$$

which proves that F is a fundamental system of solutions. ■

To illustrate the possible applications of Theorem 6.7, we will consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + 4x_5 &= 0, \\ x_1 + x_2 + 2x_3 + 4x_5 &= 0, \\ x_1 + x_2 + 2x_4 + 4x_5 &= 0, \\ x_1 + x_2 - x_3 + 3x_4 + 4x_5 &= 0. \end{aligned}$$

The rank of the system matrix is 2 (see p. 37). The solutions $(1, 1, 1, 1, -1)$, $(0, 2, 1, 1, -1)$, and $(0, 0, 2, 2, -1)$ form a fundamental system of solutions, since

$$\text{rank}\begin{Vmatrix} 1 & 1 & 1 & 1 & -1 \\ 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -1 \end{Vmatrix} = 3.$$

Exercises

1. Find a fundamental system of solutions to the following system:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= 0, \\ x_2 + 2x_3 + 2x_4 + 6x_5 &= 0, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 &= 0. \end{aligned}$$

2. Will the following systems of solutions to the system of linear equations of Exercise 1 be fundamental: (a) $\{(3, -2, -1, -1, 1), (2, 0, -2, -1, 1)\}$; (b) $\{(3, -2, -1, -1, 1), (1, -2, 1, 0, 0), (2, 0, -2, -1, 1)\}$; (c) $\{(1, -2, 1, 0, 0), (0, 0, -1, 1, 0), (4, 0, 0, -6, 2)\}$; (d) $\{(1, -2, 1, 0, 0), (1, -2, 0, 1, 0), (0, 0, 1, -1, 0), (1, -2, 3, -2, 0)\}$?

3. Find a fundamental system of solutions to the following system of linear equations:

$$\begin{aligned}
\lambda_1 b_1 x_1 + \dots + \lambda_n b_1 x_n + a_1 x_{n+1} &= 0,\\
\lambda_1 b_2 x_1 + \dots + \lambda_n b_2 x_n + a_2 x_{n+1} &= 0,\\
\dots\dots\dots\dots\dots\dots&\dots\\
\lambda_1 b_m x_1 + \dots + \lambda_n b_m x_n + a_m x_{n+1} &= 0.
\end{aligned}$$

Hint. First prove that $\left(\text{rank} \begin{Vmatrix} a & b \\ c & d \end{Vmatrix}\right) = 2$ if and only if $ad - bc \neq 0$.

4. Find a system of homogeneous linear equations for which the following rows form a fundamental system of solutions: (a) (1, 2, —1, 0, 1) and (1, 3, 1, 1, 2); (b) (2, 1, 0, 0), (1, 0, 1, 0), and (—1, 0, 0, 1), (c) (1, 2, 3, —1).

5. Let A^* be a matrix whose rows are the columns of another matrix, A. Prove that

(a) If A is a singular matrix, A^* is also singular. *Hint.* Use Theorem 6.4 and Exercise 10 in Section 4.

(b) (rank A) = (rank A^*). *Hint.* Use the result of (a).

6. Prove that the rank of a matrix remains the same under elementary transformations of columns of the first and second kinds. Is the rank of a matrix employed in elementary transformations of the third kind?

ANSWERS

Section 1

1.

(a)
$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 40,\\
x_1 + x_2 + x_3 - x_4 &= 20;
\end{aligned}$$

(b)
$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 20,\\
5x_1 + 10x_2 + 15x_3 + 20x_4 &= 200;
\end{aligned}$$

(c)
$$\begin{aligned}
x_1 + x_2 + x_3 &= 1,\\
x_1 + x_2 + x_4 &= 1,\\
x_1 + x_3 + x_4 &= 1,\\
x_2 + x_3 + x_4 &= 1,
\end{aligned}$$

(d)
$$\begin{aligned}
x_1 + x_2 &= 1,\\
x_1 + x_3 &= 1,\\
x_1 + x_4 &= 1,\\
x_2 + x_3 &= 1,\\
x_2 + x_4 &= 1,\\
x_3 + x_4 &= 1;
\end{aligned}$$

(e)
$$\begin{aligned}
20x_1 + 12x_2 + 10x_4 &= 120,\\
20x_1 + 15x_3 + 10x_4 &= 120,\\
12x_2 + 15x_3 + 10x_4 &= 120;
\end{aligned}$$

(f) $$\begin{aligned} x_1 + x_2 &= 0,\\ x_2 + x_3 &= 0,\\ \cdots&\cdots\\ x_{n-1} + x_n &= 0,\\ x_1 + x_n &= 0. \end{aligned}$$

Section 2

1. For example, $\bar{a} = (0, 1, 2, 0)$, $\bar{b} = (0, -1, -1, 2)$.
2. $(0, -3, 3, 2, 2)$. 5. $\bar{x} = (0, 1, 2, -2)$, $\bar{y} = (20/3, -25/3, 10/3, -5/3)$. 6. (a) $(1, 4, -7, 7)$.

7. (a) $\begin{Vmatrix} 1 & 1 & 1 & 1 & 40 \\ 0 & 0 & 0 & -2 & -20 \end{Vmatrix}$; (b) $\begin{Vmatrix} 1 & 1 & 1 & 1 & 20 \\ 0 & 5 & 10 & 15 & 100 \end{Vmatrix}$;

(c) $\begin{Vmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \end{Vmatrix}$; (d) $\begin{Vmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$;

(e) $\begin{Vmatrix} 20 & 12 & 0 & 10 & 120 \\ 0 & -12 & 15 & 0 & 0 \\ 0 & 0 & 30 & 10 & 120 \end{Vmatrix}$;

(f) $\begin{Vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{Vmatrix}$ for even values of n;

$\begin{Vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2 \end{Vmatrix}$ for odd values of n.

Remark. The answers to Exercises 7(a)-7(f) may differ from those given above, but the leaders must be in the same columns (see Exercise 13).

9. For example, $\begin{Vmatrix} 1 & 1 \\ 0 & 0 \end{Vmatrix}$.

Section 3

1. (a) $x_1=\frac{1}{4}(8-x_3-9x_4)$, $x_2=-\frac{1}{4}(5x_3+x_4)$; (b) $x_1=\frac{2}{3}x_2-\frac{1}{3}$, $x_3=x_4=0$; (c) the system is not consistent; (d) $x_1=\frac{1}{4}(1-3x_2-x_3)$, $x_4=1$; (e) $x_1=-\frac{1}{12}(7-18x_2+x_4)$, $x_3=\frac{1}{6}(1-5x_4)$; (f) $x_1=x_2=x_3=x_4=0$; (g) $x_1=x_2=x_3=x_4=x_5=x_6=0$; (h) $x_1=x_4-x_5$, $x_2=x_4-x_6$, $x_3=x_4$; (i) $x_1=x_4=0$, $x_2=\frac{1}{3}(x_3-2x_5)$; (j) $x_1=-3x_3-5x_5$, $x_2=2x_3+3x_5$, $x_4=0$; (k) if $n=3k$ or $n=3k+1$, there is only a zero solution, but if $n=3k+2$, then

$$x_i=\begin{cases} 0 & \text{if } i=3m, \\ -x_n & \text{if } i=3m+1, \\ x_n & \text{if } i=3m+2. \end{cases}$$

Remark. The answers to Exercises 1(a), 1(b), 1(d), 1(e), 1(h), 1(i), 1(j), and 1(k) allow for another notation for $n=3k+2$.

2. One of the sides is 10-m long, while the sum of the other three sides is 30 m. For example, 8 m, 15 m, 7 m, and 10 m. (b) If x_1 is the number of 5-kopeck coins, x_2 is the number of 10-kopeck coins, x_3 is the number of 15-kopeck coins, and x_4 is the number of 20-kopeck coins, then $x_1=x_3+2x_4$ and $x_2=20-2x_3-3x_4$. One solution is (7, 8, 3, 2). For a maximal number of 20-kopeck coins we have (12, 2, 0, 6) or (13, 0, 1, 6), while for a maximal number of 15-kopeck coins we have (10, 0, 10, 0), (c) $x_1=x_2=x_3=x_4=1/3$. (d) $x_1=x_2=x_3=x_4=1/2$. (e) If x_1, x_2, x_3, and x_4 are the number of working hours per day for the mixers with a productive capacity of 20, 12, 15, and 10 tons of concrete per hour, respectively, then $x_1=\frac{1}{4}(12-x_4)$, $x_2=\frac{1}{12}(60-5x_4)$, $x_3=\frac{1}{3}(12-x_4)$. One solution is (5/2, 25/6, 10/3, 2). For maximal working hours of the fourth mixer we have (1/4, 5/12, 1/3, 11), while for minimal working hours of the fourth mixer we have (11/4, 55/12, 11/3, 1). (f) If n is odd, all the numbers are zeros, while if n is even, all the numbers are equal in absolute value, with half of them being positive or zero and the other half negative or zero.

4. One system is

$$\begin{aligned} -2x_1-2x_2+2x_3+3x_4&=1. \\ -2x_1-x_2+x_3+2x_4&=0. \end{aligned}$$

7. No, there is either only one solution or an infinitude of solutions.

Section 4

1. (a) 2; (b) 2; (c) 4; (d) 4; (e) 3; (f) $n-1$ if n is even and n if n is odd. 2. (a) 2; (b) 3; (c) 6; (d) 5; (e) if $a_1 = \ldots = a_n$, then the rank is zero for $a_1 = -b_1 = \ldots = -b_n$ and 1 otherwise; but if there is an $a_i \neq a_1$, the rank is 1 for $b_1 = \ldots = b_n$ and 2 otherwise.

Section 5

1. For Exercise 1 in Section: (a) any two; (b) x_1, x_3, x_4 or x_2, x_3, x_4; (c) the system is not consistent; (d) x_1 and x_4 or x_2 and x_4 or x_3 and x_4; (e) x_1 and x_3 or x_1 and x_4 or x_2 and x_3 or x_2 and x_4 or x_3 and x_4; (f) all; (g) all; (h) x_1, x_2, x_3 or x_1, x_2, x_4 or x_2, x_3, x_5 or x_2, x_4, x_5 or x_3, x_5, x_6 or x_4, x_5, x_6; (i) x_1, x_2, x_4 or x_1, x_3, x_4 or x_1, x_4, x_5; (j) any three among which there is x_4; (k) all unknowns if $n = 3k$ or $n = 3k+1$, and $x_3, x_6, \ldots, x_{3k}$ and any $k+1$ of the remaining unknowns if $n = 3k+2$. For Exercise 2 in Section 3: (a) x_1 and x_4 or x_2 and x_4 or x_3 and x_4; (b) any two; (c) all; (d) all; (e) any three; (f) all if n is odd and any $n-1$ unknowns if n is even.

4. For Exercise 1 in Section 3: (h) $(1, 1, 1, 1, 0, 0)$, $(-1, 0, 0, 0, 1, 0)$, $(0, -1, 0, 0, 0, 1)$; (i) $(0, 1, 3, 0, 0)$, $(0, -2, 0, 0, 3)$; (j) $(-3, 2, 1, 0, 0)$, $(-5, 3, 0, 0, 1)$; (k) if $n = 3k$ or $n = 3k+1$, there is only a zero solution, while if $n = 3k+2$, the solution is $(-1, 1, 0, -1, 1, 0, \ldots, -1, 1, 0, -1, 1)$.

5. If n is odd, there is only a zero solution, while if n is even the solution is $(1, -1, 1, -1, \ldots, -1, 1, -1)$.

Remark. The answers to Exercises 4 and 5 may differ from those given above, but the number of rows in the established system of solutions must coincide with the number of rows in the above answers.

6. (a) $(2, 0, 0, 0) + \lambda(-1, -5, 4, 0) + \mu(-9, -1, 0, 4)$; (b) $(-1, -1, 0, 0) + \lambda(2, 3, 0, 0)$; (c) the system is not consistent; (d) $(1, -1, 0, 1) + \lambda(-3, 4, 0, 0) + \mu(-1, 0, 4, 0)$; (e) $(1, 1, 1, -1) + \lambda(-1, 0, -10, 12) + \mu(3, 2, 0, 0)$.

7. (a) $(10, 10, 10, 10) + \lambda(-1, 0, 1, 0) + \mu(0, -1, 1, 0)$; (b) $(7, 8, 3, 2) + \lambda(1, -2, 1, 0) + \mu(2, -3, 0, 1)$; (e) $(0, 0, 0, 12) + \lambda(3, 5, 4, -12)$.

Remark. The answers to Exercises 6 and 7 may differ from those given above, but the number of rows in the established system of solutions must coincide with the number of rows in the above answers.

Section 6

1. For example, $\{(1, -2, 1, 0, 0), (1, -2, 0, 1, 0), (5, -6, 0, 0, 1)\}$.

2. (a) no; (b) no; (c) yes; (d) no.

3. If all the coefficients are zeros, then $\{\bar{e}_1, \ldots, \bar{e}_n\}$, with $\bar{e}_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0)$, is a fundamental system of solutions; if there are nonzero coefficients but $\lambda_i a_j b_k - \lambda_i a_k b_j = 0$ for all, i, j, and k, then $\{a_j\bar{e}_1 - \lambda_1 b_j \bar{e}_{n+1}, a_j\bar{e}_2 - \lambda_2 b_j \bar{e}_{n+1}, \ldots, a_j\bar{e}_n -$

$\lambda_n b_j \bar{e}_{n+1}\}$ is a fundamental system if $a_j \neq 0$ and $\{\lambda_i b_k \bar{e}_1 - \lambda_1 b_k \bar{e}_i, \ldots, \lambda_i b_k \bar{e}_{i-1} - \lambda_{i-1} b_k \bar{e}_i, \lambda_i b_k \bar{e}_{i+1} - \lambda_{i+1} b_k \bar{e}_i, \ldots, \lambda_i b_k \bar{e}_n - \lambda_n b_k \bar{e}_i, \bar{e}_{n+1}\}$ is a fundamental system if $a_1 = \ldots = a_m = 0$ but $\lambda_i b_k \neq 0$, if $\Delta = \lambda_i a_j b_k - \lambda_i a_k b_j \neq 0$ for some i, j, k, then $\{\Delta \bar{e}_1 + \lambda_1 \omega \bar{e}_i, \ldots, \Delta \bar{e}_{i-1} + \lambda_{i-1} \omega \bar{e}_i, \Delta \bar{e}_{i+1} + \lambda_{i+1} \omega \bar{e}_i, \ldots, \ldots, \Delta \bar{e}_n + \lambda_n \omega \bar{e}_i\}$, with $\omega = a_k b_j - a_j b_k$, is a fundamental system.

4. (a)
$$\begin{aligned} 8x_1 - 4x_2 + x_3 + x_4 + x_5 &= 0,\\ x_2 + x_3 - 2x_4 - x_5 &= 0,\\ x_3 - 3x_4 + x_5 &= 0; \end{aligned}$$

(b)
$$x_1 - 2x_2 - x_3 + x_4 = 0;$$

(c)
$$\begin{aligned} -5x_1 + x_2 + x_3 &= 0,\\ x_2 + x_3 + 5x_4 &= 0,\\ x_3 + 3x_4 &= 0. \end{aligned}$$

Remark. The answers differ from those given above, but the rank of the systems must coincide with that of the above systems.

SOLUTIONS

Section 2

7. (f) To the last row of the given matrix we add all the preceding even-numbered rows and all the preceding odd-numbered rows multiplied by -1. Then for n even and n odd we get the step-like matrices

$$\left\|\begin{matrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{matrix}\right\| \quad \text{and} \quad \left\|\begin{matrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 2 \end{matrix}\right\|,$$

respectively.

11. For $a \neq 0$ the problem is solved by obtaining the following chain of matrices successively:

$$\left\|\begin{matrix} a & 0\\ 0 & b \end{matrix}\right\|, \left\|\begin{matrix} a & 0\\ a & b \end{matrix}\right\|, \left\|\begin{matrix} a & 0\\ a+1 & b \end{matrix}\right\|, \left\|\begin{matrix} a & 0\\ 1 & b \end{matrix}\right\|, \left\|\begin{matrix} a & -ab\\ 1 & 0 \end{matrix}\right\|, \left\|\begin{matrix} 0 & -ab\\ 1 & 0 \end{matrix}\right\|,$$

$$\left\|\begin{matrix} 1 & 0\\ 0 & -ab \end{matrix}\right\|,$$

12. Note that when we perform a single elementary transformation, each row of the new matrix is a linear combination of the rows of the given matrix. After this, use the results of Exercise 6(b) several times.

13. Suppose that $\bar{b}_1, \ldots, \bar{b}_r$ and $\bar{c}_1, \ldots, \bar{c}_s$ are the nonzero rows of matrices B and C, respectively. Let us assume that the leaders of these rows are in the columns with numbers $k_1, \ldots, k_r$ and $l_1, \ldots, l_s$, where $k_1 < \ldots < k_r$ and $l_1 < \ldots < l_s$. From Theorem 2.4 it follows that going over from B to C as well as from C to B requires elementary transformations. According to Exercise 12, each of the $\bar{b}_i$ is a linear combination of the rows $\bar{c}_1, \ldots, \bar{c}_s$, while each of the $\bar{c}_j$ is a linear combination of the rows $\bar{b}_1, \ldots, \bar{b}_r$. This all yields $k_1 = l_1$. Next, if $\bar{b}_2 = \xi_1\bar{c}_1 + \ldots + \xi_s\bar{c}_s$ and $\bar{c}_2 = \eta_1\bar{b}_1 + \ldots + \eta_r\bar{b}_r$, then $\xi_1 = 0 = \eta_1$ and $k_2 = l_2$, just as in the case above. Considering similar expressions for $\bar{b}_3$ and $\bar{c}_3$, we see that $k_3 = l_3$. If $r \leqslant s$, then, continuing this process, we get $k_i = l_i$, where $1 \leqslant i \leqslant r$. For $r < s$, the condition $\bar{c}_{r+1} = \zeta_1\bar{b}_1 + \ldots + \zeta_r\bar{b}_r$ implies $\zeta_1 = \ldots = \zeta_r = 0$, which means that $\bar{0} \neq \bar{c}_{r+1} = \bar{0}$, which is impossible. The case where $s \leqslant r$ can be considered in a similar manner.

Section 3

1. (k) The system matrix has the form

$$\left\|\begin{array}{ccc|cccccccc}
1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1
\end{array}\right\|.$$

We subtract the first row from the second, interchange the second and third rows, and subtract the (old) second row from the fourth. The result is the matrix

$$\left\|\begin{array}{ccc|ccc|cc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \ldots \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & \ldots \\
\hline
\cdot & \cdot\;\cdot & \cdot & \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot & \cdot\;\cdot
\end{array}\right\|.$$

Now we subtract the fourth row from the fifth, interchange the fifth and sixth rows, and subtract the (old) fifth row from the seventh. The result is the matrix

$$\left\| \begin{array}{ccc|ccc|ccc|cc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ \hline \multicolumn{11}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \end{array} \right\| .$$

Next, from the eighth row we subtract the seventh, interchange the eighth and tenth rows, and subtract the (old) eighth row from the tenth. Continuing this process, for $n = 3k$, $n = 3k + 2$, and $n = 3k + 1$ we arrive at the matrices

$$\left\| \begin{array}{ccc|ccc} \multicolumn{3}{c|}{B} & \multicolumn{3}{c}{O} \\ \hline 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 & 1 \end{array} \right\|, \qquad \left\| \begin{array}{ccc|cc} \multicolumn{3}{c|}{B} & \multicolumn{2}{c}{O} \\ \hline 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{array} \right\|,$$

and

$$\left\| \begin{array}{ccc|c} \multicolumn{3}{c|}{B} & O \\ \hline 0 & \dots & 0 & 1 \end{array} \right\|,$$

respectively, with O the null matrix and

$$B = \left\| \begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline \multicolumn{3}{c|}{\dots} & \multicolumn{3}{c|}{\dots} & \multicolumn{3}{c|}{\dots} & \multicolumn{3}{c}{\dots} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{array} \right\| .$$

In the first two cases, after obvious elementary transformations we get

$$\left\|\begin{array}{ccc|ccc} & B & & & O & \\ \hline 0 & \dots & 0 & 1 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{array}\right\| \quad \text{and} \quad \left\|\begin{array}{ccc|cc} & B & & & O \\ \hline 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right\|,$$

respectively. Now it is clear that in the first and third cases the system has only a zero solution. In the second case we get $x_{3i} = 0$ and $x_{3i+2} = -x_{3i+1} = x_n$ (note that $n = 3k + 2$).

5. If row $\bar{b}$, of which we are speaking here, can be represented in the form $\bar{b} = \lambda_1\bar{b}_1 + \dots + \lambda_n\bar{b}_n$, where $\bar{b}_1, \dots, \bar{b}_n$ are the other rows, adding the linear combination $-\lambda_1\bar{b}_1 - \dots - \lambda_n\bar{b}_n$ to $\bar{b}$ we transform this row into a zero row. Now employ Theorems 1.1 and 3.2.

6. The augmented matrix must be a null matrix. It is obvious that this condition is sufficient. To prove that it is necessary we employ the fact that the rows $\bar{0}$ and $\bar{e}_i = (\underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0)$ $(i = 1, 2, \dots, n)$ are solutions.

Section 4

2. (e) It is easy to go, via elementary transformations, from the given matrix to the matrix

$$\left\|\begin{array}{cccc} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_2-a_1 & a_2-a_1 & \dots & a_2-a_1 \\ \dots & \dots & \dots & \dots \\ a_n-a_1 & a_n-a_1 & \dots & a_n-a_1 \end{array}\right\|.$$

If $a_1 = \dots = a_n$, then all the rows except the first, are zero rows, and the rank is equal to 0 or 1 depending on whether the first row is a zero row or not. If $a_i \neq a_1$ for a certain i, the ith row is nonzero. Through obvious elementary transformations the above matrix can be transformed into

$$\left\|\begin{array}{cccc} a_1+b_1 & a_1+b_2 & \dots & a_1+b_n \\ a_i-a_1 & a_i-a_1 & \dots & a_i-a_1 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{array}\right\|.$$

If we multiply the second row by $-(a_1 + b_1)/(a_i - a_1)$ and add the product to the first row, we obtain a matrix that for $b_1 = \dots = b_n$ has only one nonzero row; otherwise, if we interchange the first two rows, we arrive at a step-like matrix with two nonzero rows.

5. If to each row of matrix A we add the corresponding row of matrix B, we arrive at a matrix C that coincides with the matrix on the left-hand side of the inequality we are proving here. Therefore, allowing for Theorem 4.6 and Exercise 3, we get

$$(\operatorname{rank} C) \leqslant \left(\operatorname{rank} \left\| \begin{array}{c} C \\ \hline B \end{array} \right\| \right) = \left(\operatorname{rank} \left\| \begin{array}{c} A \\ \hline B \end{array} \right\| \right)$$

$$\leqslant (\operatorname{rank} A) + (\operatorname{rank} B).$$

10. It is sufficient to note that if the ith column is a linear combination of the other rows with coefficients $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n$, then the row $(\lambda_1, \ldots, \lambda_{i-1}, -1, \lambda_{i+1}, \ldots, \lambda_n)$ serves as a nonzero solution to the system of homogeneous linear equations with matrix A. Then we need only employ Theorem 4.1.

Section 5

3. If the augmented matrix of the system of linear equations proves to be nonsingular, then the rank of this matrix is equal to n, while the rank of the system matrix does not exceed $n - 1$, which in view of Theorem 5.1 means that the system of linear equations is not consistent.

Section 6

3. If $ab - cd \neq 0$, then for $a \neq 0$ it is sufficient to go over from $\begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$ to $\begin{Vmatrix} a & b \\ 0 & d-(bc)/a \end{Vmatrix}$ by adding the product of the first row by $-c/a$ to the second row (note that $d - (bc)/a \neq 0$). But if $a = 0$, then $c \neq 0$, and we can proceed in a similar manner. If the rank of $\begin{Vmatrix} a & b \\ c & d \end{Vmatrix}$ is 2, then $a \neq 0$ or $c \neq 0$. For $a \neq 0$, carrying out the same transformation as in the above case, we obtain $d - (bc)/a \neq 0$, whence $ad - bc \neq 0$. For $c \neq 0$ we can proceed in a similar manner.

6. Since elementary transformations of the columns of a matrix A correspond to elementary transformations of the rows of the matrix A^*, it is sufficient to allow for Exercise 9(a) and Theorem 4.6.

The book contains a complete exposition of the theory of systems of linear equations employing only elementary operations on matrices. The method of complete mathematical induction is, formally, not used here. However, in some cases it is hidden behind the words "etc.". Each section is followed by exercises. The main purpose of the exercises is to give the reader an opportunity to test his mastery of the material. The book is intended for a wide circle of readers, including pupils of senior classes of secondary schools, who are interested in mathematics.

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